Special Holonomy in Gauge Theory

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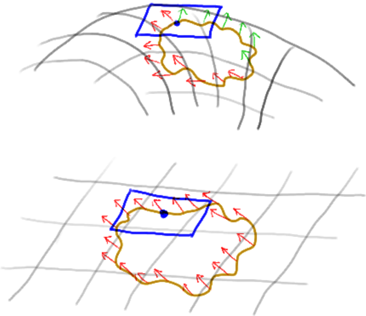
Holonomy I

- Basic idea: Parallel transport of a tangent vector along a closed curve induces a linear automorphism of the tangent space at $p \in M$:



$$\mathrm{Hol}_p(\gamma) : \mathrm{T}_p M \xrightarrow{\cong} \mathrm{T}_p M$$

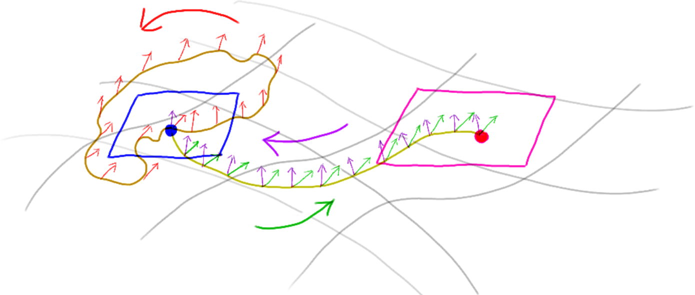
$$\rightsquigarrow \quad \mathrm{Hol}_p(\gamma) \in \mathrm{Aut}(\mathrm{T}_p M) \cong \mathrm{GL}(n)$$

This depends on the curve γ and the base point $p \in M$.

- The collection of all such holonomy isomorphisms for each curve γ determines a subgroup $\mathrm{Hol}_p \subset \mathrm{GL}(n)$.

Holonomy II

- Given another point $q \in M$, one can identify the holonomy automorphisms of p and q via parallel transport.



$\Rightarrow$ The holonomy subgroup Hol(g) only depends on the metric g!

- Question: **Which subgroups of SO(n) arise as holonomy groups?**

Berger's classification

- Holonomy classification theorem by Berger (1955):

		Kähler	Calabi-Yau	Hyper-kähler	quat. Kähler	G_2-hol.	Spin(7)-hol.
	SO(n)	**U($\frac{n}{2}$)**	**SU($\frac{n}{2}$)**	**Sp($\frac{n}{4}$)**	**Sp(1)Sp($\frac{n}{4}$)**	**G_2**	**Spin(7)**
$n=2$:	✓	✓					
$n=3$:	✓						
$n=4$:	✓	✓	✓(2)				
$n=5$:	✓						
$n=6$:	✓	✓	✓(2)				
$n=7$:	✓					✓(1)	
$n=8$:	✓	✓	✓(2)	✓(3)	✓		✓(1)
$n=4m+1$:	✓						
$n=4m+2$:	✓	✓	✓(2)				for
$n=4m+3$:	✓						$m \geq 2$
$n=4m+4$:	✓	✓	✓(2)	✓($m+2$)	✓		

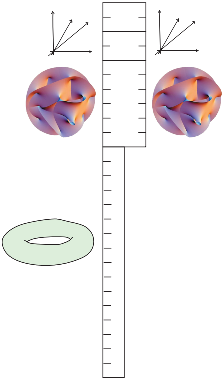
- Ricci-flatness (red cases) is related to the existence of globally defined, covariantly constant spinors. $\Rightarrow$ numbers in parenthesis

Special holonomy in compactification

- Classical usage: The equations for (partially) unbroken 4d SUSY after a Kaluza-Klein compactification reduce to the internal space admitting a parallel spinor, i.e. the internal space being Ricci-flat. (Candelas et al. 1985)

  $\rightsquigarrow$ Calabi-Yau compactification

- More recently (following the same basic idea):
  - G_2-compactification of 10d string theory
    $\Rightarrow$ 3d $N = 1$ toy models
  - G_2-compactification of 11d M-theory
    $\Rightarrow$ 4d $N = 1$ models
  - Spin(7)-compactification of 11d M-theory
    $\Rightarrow$ 3d $N = 1$ toy models



- However, spaces with special holonomy can also be used in higher dimensional gauge theory.

Self-duality in 4d Yang-Mills theory I

- A gauge field strength $F_{\mu\nu}$ corresponds to a certain Lie-algebra-valued curvature 2-form $F \in \Omega^2(M; \mathrm{ad}\, P)$.
- On an oriented Riemannian manifold there is a splitting

$$\Omega^2(M) = \Omega^2_+ \oplus \Omega^2_-$$

into the ± 1-eigenspaces of the Hodge operator

$$\star : \Omega^k(M) \xrightarrow{\cong} \Omega^{4-k}(M).$$

- Corresponding splitting of the gauge field strength: $F = F_+ + F_-$

- Define the topological term

$$C := \int_M \mathrm{Tr}(F \wedge F) = \int_M \|F_+\|^2 - \int_M \|F_-\|^2 = 4\pi^2 \overbrace{p_1(M)}^{\text{instanton number}}$$

- The 4d pure Yang-Mills action

$$S_{\mathrm{YM}} := \int_M \mathrm{Tr}(\star F \wedge F) = \int_M \|F\|^2 = \int_M \|F_+\|^2 + \int_M \|F_-\|^2$$
$$= C + 2\int_M \|F_-\|^2 = -C + 2\int_M \|F_+\|^2$$

is then minimized, if either $F_+ = 0$ (anti-selfdual) or $F_- = 0$ (selfdual). Such solutions are called instantons.

$$\rightsquigarrow \text{ in components: } \begin{cases} F_{12} = \pm F_{34} \\ F_{13} = \pm F_{42} \\ F_{14} = \pm F_{23} \end{cases}$$

Example: Anti-self-duality of 8d Yang-Mills theory I

- Question: **Can such a description of instantons be generalized to higher dimensions?**
  $\Rightarrow$ Yes, using special holonomy! (Donaldson et al. 1998)
- Example: **8d Yang-Mills theory on a Spin(7)-manifold X.**
- The Spin(7)-holonomy structure can be represented by a certain Spin(7)-invariant 4-form $\Phi \in \Omega^4(X)$.
- This allows to define a duality operator

$$\star_\Phi := \star(\Phi \wedge \mathrm{Id}) : \Omega^k(X) \xrightarrow{\cong} \Omega^{4-k}(X),$$

which induces a corresponding eigenspace splitting

$$\Omega^2(X) = \Omega^2_7 \oplus \Omega^2_{21}$$

with subscripts indicating the dimension of the respective eigenspace.

- The 8d pure Yang-Mills action on a Spin(7)-manifold

$$S_{\mathrm{YM}} = \int_X \|F\|^2 = \underbrace{\int_X \Omega \wedge \mathrm{Tr}(F \wedge F)}_{\text{topological term}} + \int_X \|F_7\|^2$$

  is minimized for $F_7 = 0$, which is called the Spin(7) instanton equation. $\Rightarrow$ defines Spin(7) instantons.
- Similar statements can be developed for G_2-manifolds and Calabi-Yau 4-folds.

Dimensional reduction of 4d instantons

- Consider the 4d ASD instanton equations $F_{ij} = -\epsilon_{ijkl} F^{kl}$, where each gauge field-strength/curvature component can be written as $F_{ij} = [\nabla_i, \nabla_j]$.
- Drop the dependency of the coordinate x_4 and replace ∇_4 by a function $\phi = \phi(x_1, x_2, x_3)$.

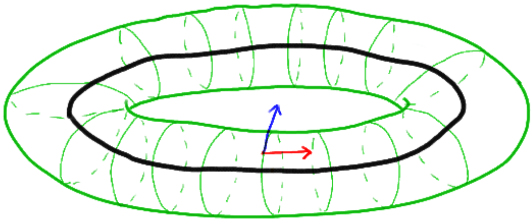
$$\Rightarrow \left.\begin{array}{l} F_{12} = [\nabla_1, \nabla_2] = -[\nabla_3, \phi] = -F_{34} \\ F_{13} = [\nabla_1, \nabla_3] = [\nabla_2, \phi] = -F_{42} \\ F_{14} = [\nabla_1, \phi] = -[\nabla_2, \nabla_3] = -F_{23} \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} \tilde{F}_{12} = -\nabla_3 \phi \\ \tilde{F}_{13} = \nabla_2 \phi \\ \tilde{F}_{23} = -\nabla_1 \phi \end{array}\right\},$$

$$\Longleftrightarrow \quad \tilde{F} = \star \mathrm{d}\phi$$

where $\tilde{F}$ is the 3d field-strength/curvature consisting of the components F_{12}, F_{13} and F_{23}. The result is called the Bogomolny or monopole equation.

Dimensional reduction of 8d Spin(7) instantons

- The 8d total space of the positive spinor bundle $S^+(M)$ of a Calabi-Yau 2-fold is a Spin(7) manifold.
  $\Rightarrow$ This Spin(7) manifold locally looks like $\mathbb{R}^4 \times S^+(\mathbb{R}^4)$



- By dropping the dependency in the vertical $S^+(\mathbb{R}^4)$-directions, the Spin(7) instanton equations are dimensionally reduced to the 4d Seiberg-Witten equations:

$$\begin{cases} F_+ = [\Phi, \Phi^*] \\ \not{D}\Phi = 0 \end{cases}$$

Summary

- Spaces with special holonomy are used in space-time compactifications where a certain amount of SUSY remains unbroken.
  $\Rightarrow$ Important for chiral matter, hierarchy problem, etc.
- Special holonomy allows to generalize the concept of selfduality to higher dimensions.
- Dimensional reduction of instanton equations connects various branches of gauge theory.

- Currently working on the dim. reduction of Spin(7) instantons to 6d.
  $\rightsquigarrow$ Donaldson-Uhlenbeck-Yau equations

For further reading

- Introductions to special holonomy:
  - Salamon: *Riemannian geometry and holonomy groups*, 1989.
  - Joyce: *Riemannian Holonomy Groups and Calibrated Geometry*, 2007.
- Calabi-Yau compactification:
  - Candelas et al.: Nucl. Phys. B258 (1985), 46-74.
- Higher dimensional gauge theory and dimensional reduction:
  - Donaldson et al.: *Gauge theory in higher dimensions*, In: The Geometric Universe, 1998.
  - Hitchin: Proc. London Math. Soc. 55 (1987), 59-126.
  - Baulieu et al.: hep-th/9704167