Special Holonomy in Gauge Theory

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Holonomy I

- Basic idea: Parallel transport of a tangent vector along a closed curve induces a linear automorphism of the tangent space at \( p \in M \):

\[
\text{Hol}_p(\gamma) : T_pM \xrightarrow{\sim} T_pM
\]

\[
\sim \quad \text{Hol}_p(\gamma) \in \text{Aut}(T_pM) \cong \text{GL}(n)
\]

This depends on the curve \( \gamma \) and the base point \( p \in M \).

- The collection of all such holonomy isomorphisms for each curve \( \gamma \) determines a subgroup \( \text{Hol}_p \subset \text{GL}(n) \).
Given another point $q \in M$, one can identify the holonomy automorphisms of $p$ and $q$ via parallel transport.

⇒ The holonomy subgroup $\text{Hol}(g)$ only depends on the metric $g$!

Question: Which subgroups of $\text{SO}(n)$ arise as holonomy groups?
Berger’s classification

Holonomy classification theorem by Berger (1955):

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<tbody>
<tr>
<td>SO(n)</td>
<td>U((\frac{n}{2}))</td>
<td>SU((\frac{n}{2}))</td>
<td>Sp((\frac{n}{4}))</td>
<td>Sp(1)Sp((\frac{n}{4}))</td>
<td>G₂</td>
<td>Spin(7)</td>
</tr>
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- \(n = 2\): ✓ ✓
- \(n = 3\): ✓
- \(n = 4\): ✓ ✓ ✓ (2)
- \(n = 5\): ✓
- \(n = 6\): ✓ ✓ ✓ (2)
- \(n = 7\): ✓ ✓ ✓ (2) ✓ (1)
- \(n = 8\): ✓ ✓ ✓ (2) ✓ (3) ✓ ✓ (1)

- \(n = 4m + 1\): ✓
- \(n = 4m + 2\): ✓ ✓ ✓ (2) for \(m \geq 2\)
- \(n = 4m + 3\): ✓
- \(n = 4m + 4\): ✓ ✓ ✓ (2) ✓ (m + 2) ✓

Ricci-flatness (red cases) is related to the existence of globally defined, covariantly constant spinors. ⇒ numbers in parenthesis
Special holonomy in compactification

- Classical usage: The equations for (partially) unbroken 4d SUSY after a Kaluza-Klein compactification reduce to the internal space admitting a parallel spinor, i.e. the internal space being Ricci-flat. (Candelas et al. 1985)
  \[\Rightarrow\] Calabi-Yau compactification

- More recently (following the same basic idea):
  - $G_2$-compactification of 10d string theory
    \[\Rightarrow\] 3d $N = 1$ toy models
  - $G_2$-compactification of 11d M-theory
    \[\Rightarrow\] 4d $N = 1$ models
  - Spin(7)-compactification of 11d M-theory
    \[\Rightarrow\] 3d $N = 1$ toy models

- However, spaces with special holonomy can also be used in higher dimensional gauge theory.
A gauge field strength $F_{\mu\nu}$ corresponds to a certain Lie-algebra-valued curvature 2-form $F \in \Omega^2(M; \text{ad } P)$.

On an oriented Riemannian manifold there is a splitting

$$\Omega^2(M) = \Omega^2_+ \oplus \Omega^2_-$$

into the $\pm 1$-eigenspaces of the Hodge operator

$$\star : \Omega^k(M) \xrightarrow{\cong} \Omega^{4-k}(M).$$

Corresponding splitting of the gauge field strength: $F = F_+ + F_-$
Define the topological term

\[ C := \int_M \text{Tr}(F \wedge F) = \int_M \|F_+\|^2 - \int_M \|F_-\|^2 = 4\pi^2 \hat{p}_1(M) \]

The 4d pure Yang-Mills action

\[ S_{YM} := \int_M \text{Tr}(\star F \wedge F) = \int_M \|F\|^2 = \int_M \|F_+\|^2 + \int_M \|F_-\|^2 \]

\[ = C + 2 \int_M \|F_-\|^2 = -C + 2 \int_M \|F_+\|^2 \]

is then minimized, if either \( F_+ = 0 \) (anti-selfdual) or \( F_- = 0 \) (selfdual). Such solutions are called instantons.

\[ \rightarrow \text{in components:} \begin{cases} F_{12} = \pm F_{34} \\ F_{13} = \pm F_{42} \\ F_{14} = \pm F_{23} \end{cases} \]
Question: Can such a description of instantons be generalized to higher dimensions?  
⇒ Yes, using special holonomy! (Donaldson et al. 1998)

Example: 8d Yang-Mills theory on a Spin(7)-manifold $X$.

The Spin(7)-holonomy structure can be represented by a certain Spin(7)-invariant 4-form $\Phi \in \Omega^4(X)$.

This allows to define a duality operator

\[
\star \Phi := \star (\Phi \wedge \text{Id}) : \Omega^k(X) \xrightarrow{\cong} \Omega^{4-k}(X),
\]

which induces a corresponding eigenspace splitting

\[
\Omega^2(X) = \Omega^2_7 \oplus \Omega^2_{21}
\]

with subscripts indicating the dimension of the respective eigenspace.
The 8d pure Yang-Mills action on a Spin(7)-manifold

\[ S_{YM} = \int_X \|F\|^2 = \int_X \Omega \wedge \text{Tr}(F \wedge F) + \int_X \|F_7\|^2 \]

is minimized for \( F_7 = 0 \), which is called the Spin(7) instanton equation. \( \Rightarrow \) defines Spin(7) instantons.

Similar statements can be developed for \( G_2 \)-manifolds and Calabi-Yau 4-folds.
Consider the 4d ASD instanton equations \( F_{ij} = -\epsilon_{ijkl} F^{kl} \), where each gauge field-strength/curvature component can be written as \( F_{ij} = [\nabla_i, \nabla_j] \).

Drop the dependency of the coordinate \( x_4 \) and replace \( \nabla_4 \) by a function \( \phi = \phi(x_1, x_2, x_3) \).

\[
\begin{align*}
F_{12} &= [\nabla_1, \nabla_2] = -[\nabla_3, \phi] = -F_{34} \\
F_{13} &= [\nabla_1, \nabla_3] = [\nabla_2, \phi] = -F_{42} \\
F_{14} &= [\nabla_1, \phi] = -[\nabla_2, \nabla_3] = -F_{23}
\end{align*}
\]

\[
\begin{align*}
\tilde{F}_{12} &= -\nabla_3 \phi \\
\tilde{F}_{13} &= \nabla_2 \phi \\
\tilde{F}_{23} &= -\nabla_1 \phi
\end{align*}
\]

\( \iff \) \( \tilde{F} = \star d\phi \)

where \( \tilde{F} \) is the 3d field-strength/curvature consisting of the components \( F_{12}, F_{13} \) and \( F_{23} \). The result is called the Bogomolny or monopole equation.
The 8d total space of the positive spinor bundle $S^+(M)$ of a Calabi-Yau 2-fold is a Spin(7) manifold. 

⇒ This Spin(7) manifold locally looks like $\mathbb{R}^4 \times S^+(\mathbb{R}^4)$

By dropping the dependency in the vertical $S^+(\mathbb{R}^4)$-directions, the Spin(7) instanton equations are dimensionally reduced to the 4d Seiberg-Witten equations:

\[
\begin{align*}
F_+ &= [\Phi, \Phi^*] \\
\mathcal{D}\Phi &= 0
\end{align*}
\]
Spaces with special holonomy are used in space-time compactifications where a certain amount of SUSY remains unbroken. ⇒ Important for chiral matter, hierarchy problem, etc.

Special holonomy allows to generalize the concept of selfduality to higher dimensions.

Dimensional reduction of instanton equations connects various branches of gauge theory.

Currently working on the dim. reduction of Spin(7) instantons to 6d. ↦ Donaldson-Uhlenbeck-Yau equations
For further reading

- Introductions to special holonomy:

- Calabi-Yau compactification:

- Higher dimensional gauge theory and dimensional reduction:
  - Baulieu et al.: hep-th/9704167