A Computational Tool for Line Bundle Cohomology

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Contents

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- **Algorithm:** R. Blumenhagen, B. Jurke, T. Rahn, H. Roschy – arXiv:1003.5217 [hep-th]
- **Proof A:** T. Rahn, H. Roschy – arXiv:1006.2392 [hep-th]
- **Proof B:** S.-T. Jow – arXiv:1006.0780 [math.AG]

Motivation: Why line bundle cohomology?

In the context of toric geometry line bundles are appearing all over:

- Description of the **tangent bundle**:

$$0 \longrightarrow \mathcal{O}_X^{\oplus r} \overset{\alpha}{\hookrightarrow} \bigoplus_{i=1}^{n} \mathcal{O}_X(D_i) \overset{\beta}{\twoheadrightarrow} T_X \longrightarrow 0$$

- **Monad** and **extension bundle** constructions:

$$0 \longrightarrow V \overset{f}{\hookrightarrow} \bigoplus_{i=1}^{r_B} \mathcal{O}_X(b_i) \overset{g}{\twoheadrightarrow} \bigoplus_{i=1}^{r_C} \mathcal{O}_X(c_i) \longrightarrow 0$$

$$0 \longrightarrow \bigoplus_{i=1}^{r_A} \mathcal{O}_X(a_i) \overset{f}{\hookrightarrow} W \overset{g}{\twoheadrightarrow} \bigoplus_{i=1}^{r_C} \mathcal{O}_X(c_i) \longrightarrow 0$$

- **Koszul sequence** for subspaces:

$$0 \longrightarrow \mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X \twoheadrightarrow \mathcal{O}_D \longrightarrow 0$$

Motivation: Why line bundle cohomology?

From those short exact sequences one considers the **induced long exact sequences** of the sheaf cohomology:

$$0 \longrightarrow \mathcal{O}_X^{\oplus r} \hookrightarrow \bigoplus_{k=1}^{n} \mathcal{O}_X(D_k) \twoheadrightarrow T_X \longrightarrow 0$$

$$\downarrow$$

$$0 \longrightarrow H^0(X;\mathcal{O}_X)^{\oplus r} \longrightarrow \bigoplus_{k=1}^{n} H^0(X;\mathcal{O}_X(D_k)) \longrightarrow H^0(X;T_X) \longrightarrow$$

$$\longrightarrow H^1(X;\mathcal{O}_X)^{\oplus r} \longrightarrow \bigoplus_{k=1}^{n} H^1(X;\mathcal{O}_X(D_k)) \longrightarrow H^1(X;T_X) \longrightarrow$$

$$\longrightarrow H^2(X;\mathcal{O}_X)^{\oplus r} \longrightarrow \bigoplus_{k=1}^{n} H^2(X;\mathcal{O}_X(D_k)) \longrightarrow H^2(X;T_X) \longrightarrow \ldots$$

Description of the algorithm

Ultimately, we are interested in computing $\dim H^i(X; \mathcal{O}_X(D))$.

Setting:

- X toric variety
- homogeneous coordinates $H = \{x_1, \ldots, x_n\}$
- Stanley-Reisner ideal $\mathrm{SR} = \langle \mathcal{S}_1, \ldots, \mathcal{S}_N \rangle$

coords	GLSM charges Q^1	Q^2	divisor class
x_1	1	0	H
x_2	1	0	H
x_3	1	1	$H + X$
x_4	0	1	X

Take a squarefree monomial $\mathcal{Q} = x_{i_1} \cdots x_{i_k}$ of the coordinates H.
Basic idea: Count rational functions of the form

$$R^{\mathcal{Q}}(x_1, \ldots, x_n) = \frac{T(\overbrace{x_{j_1}, \ldots, x_{j_{n-k}}}^{\text{complement coordinates } H \setminus \mathcal{Q}})}{\underbrace{x_{i_1} \cdots x_{i_k}}_{\mathcal{Q}} \cdot W(\underbrace{x_{i_1}, \ldots, x_{i_k}}_{\text{coordinates in } \mathcal{Q}})}$$

T, W monomials

Description of the algorithm

Due to a surprising **vanishing result** it actually suffices to consider a restricted number of rational function:

$$\rightsquigarrow \boxed{\mathcal{N}_D(\mathcal{Q}) := \dim\left\{R^{\mathcal{Q}} : \deg R^{\mathcal{Q}} = D\right\}} \qquad \mathcal{Q} = \text{union of squarefree SR ideal generators}$$

Q: To which cohomology group H^i does this number contribute?
Need to trace back how often the same union of SR ideal generators arises.

$$S^k_\alpha := \{\mathcal{S}_{\alpha_1}, \ldots, \mathcal{S}_{\alpha_k}\} \subset \mathrm{SR} \quad \text{set of } k \text{ SR generators}$$

Let $\mathcal{Q}(S^k_\alpha)$ be the squarefree monomial from the union of those generators.

$$\boxed{N(S^k_\alpha) := \underbrace{\left|\mathcal{Q}(S^k_\alpha)\right|}_{\#(\text{individual coordinates in the union})} - \overbrace{k}^{\#(\text{unified SR generators})}}$$

$\rightsquigarrow$ Gives a measure for the multiplicity of coordinates in different SR generators, from which the union $\mathcal{Q}(S^k_\alpha)$ arises

(working through the powerset of SR generators)

Description of the algorithm

Note that $N(S_\alpha^k) \in \{-N, ..., n\}$, where $\begin{cases} N \text{ number of SR gens} \\ n \text{ number of coords.} \end{cases}$

Count the number of combinations of SR generators yielding the same monomial and multiplicity measure:

$$\dim \mathfrak{C}^i(\mathcal{Q}) := \# \left\{ S_\alpha^k \subset \mathrm{SR} : \begin{matrix} \mathcal{Q}(S_\alpha^k) = \mathcal{Q} \\ N(S_\alpha^k) = i \end{matrix} \right\}$$

Consider then the cohomology $\mathfrak{h}^i(\mathcal{Q})$ of the sequence

$$0 \longrightarrow \mathfrak{C}^{-N} \longrightarrow \mathfrak{C}^{-N+1} \longrightarrow \ldots \longrightarrow \mathfrak{C}^n \longrightarrow 0$$

In most cases those multiplicity factors will either be 0 or 1.

(In precise mathematical terms this sequence corresponds to a subcomplex of the full Taylor resolution of the Stanley-Reisner ring $\mathbb{C}[H]/\mathrm{SR}$, which defines the mappings $\mathfrak{C}^i \longrightarrow \mathfrak{C}^{i+1}$.)

Algorithm overview

Dimension of line bundle sheaf cohomology

$$\dim H^i(X;\mathcal{O}_X(D)) = \sum_{\mathcal{Q}} \underbrace{\mathfrak{h}^i(\mathcal{Q})}_{\text{multiplicity factor}} \cdot \overbrace{\mathcal{N}_D(\mathcal{Q})}^{\#(\text{rational functions})}$$

where the sum ranges over unions of SR generators

1. Determine all monomials $\mathcal{Q}$ from unions of SR gens.
2. For each such $\mathcal{Q}$ compute the corresponding numbers of SR gen. combinations $\dim \mathfrak{C}^i(\mathcal{Q})$
3. From those determine the multiplicity factors $\mathfrak{h}^i(\mathcal{Q})$
4. For each $\mathcal{Q}$ where $\mathfrak{h}^i(\mathcal{Q}) \neq 0$ count the number of rational functions $\mathcal{N}_D(\mathcal{Q})$.
5. Sum over all relevant contributions $\mathfrak{h}^i(\mathcal{Q}) \cdot \mathcal{N}_D(\mathcal{Q})$.

$$\rightsquigarrow \quad \dim H^i(X;\mathcal{O}_X(D))$$

Example: del-Pezzo-3 surface

	GLSM charges				divisor
	Q^1	Q^2	Q^3	Q^4	class
x_1	1	0	0	1	$H+Z$
x_2	1	0	1	0	$H+Y$
x_3	1	1	0	0	$H+X$
x_4	0	1	0	0	X
x_5	0	0	1	0	Y
x_6	0	0	0	1	Z

$$\mathrm{SR}(dP_3) = \langle x_1x_2,\ x_1x_3,\ x_1x_6,\ x_2x_3,\ x_2x_5,\ x_3x_4,\ x_4x_5,\ x_4x_6,\ x_5x_6 \rangle$$

Consider the monomial $\mathcal{Q} = x_1x_2x_3$:

$$\{x_1x_2,\ x_1x_3\} \qquad N = 3-2 = 1$$
$$\{x_1x_2,\ x_2x_3\} \qquad N = 3-2 = 1$$
$$\{x_1x_3,\ x_2x_3\} \qquad N = 3-2 = 1$$
$$\{x_1x_2,\ x_1x_3,\ x_2x_3\} \qquad N = 3-3 = 0$$

Yields the sequence

$$0 \longrightarrow \mathfrak{C}^0 = \mathbb{C} \longrightarrow \mathfrak{C}^1 = \mathbb{C}^3 \longrightarrow 0$$

which leads to
$\mathfrak{h}^1(\mathcal{Q}) = \mathfrak{h}^1(x_1x_2x_3) = 2$.

In total one finds:

- 46 monomials $\mathcal{Q}(S_\alpha^k)$ arise from unions of SR gens
- 34 monomials thereof have $\mathfrak{h}^i(\mathcal{Q}(S_\alpha^k)) \neq 0$ for some i

Example: del-Pezzo-3 surface

For the monomial $\mathcal{Q} = x_1 x_2 x_3$ we now have to count rational functions of the form:

$$R^{\mathcal{Q}}(x_1, \ldots, x_6) = \frac{T(x_4, x_5, x_6)}{x_1 x_2 x_3 \cdot W(x_1, x_2, x_3)}, \qquad T,\ W \text{ monomials}$$

Let $\|x_i\| \in \mathbb{N}$ denote the exponent of the corresponding coordinate.

$$\begin{aligned} \deg R^{\mathcal{Q}} = \big(- \|x_1\| - \|x_2\| - \|x_3\| - 3, \\ \|x_4\| - \|x_3\| - 1, \\ \|x_5\| - \|x_2\| - 1, \\ \|x_6\| - \|x_1\| - 1 \big) \end{aligned}$$

For $D = -4H = (-4, 0, 0, 0)$ we find the following solutions:

$$\begin{aligned} \mathcal{N}_D(\mathcal{Q}) &= \dim\{R^{\mathcal{Q}} : \deg R^{\mathcal{Q}} = -4H\} \\ &= \#\left\{ \frac{x_4 x_5 x_6^2}{x_1 x_2 x_3 \cdot x_1}, \frac{x_4 x_5^2 x_6}{x_1 x_2 x_3 \cdot x_2}, \frac{x_4^2 x_5 x_6}{x_1 x_2 x_3 \cdot x_4} \right\} = 3 \end{aligned}$$

Example: del-Pezzo-3 surface

Then due to $\overbrace{\mathfrak{h}^1(\mathcal{Q}) = 2}^{\text{multiplicity}}$ and $\overbrace{\mathcal{N}_D(\mathcal{Q}) = 3}^{\text{rational functions}}$ the contribution of the squarefree monomial $\mathcal{Q} = x_1 x_2 x_3$ to the cohomology $H^i(X; \mathcal{O}_X(D))$ is:

$$\mathfrak{h}^i(\mathcal{Q}) \cdot \mathcal{N}_D(\mathcal{Q}) \quad \rightsquigarrow \quad \begin{cases} 0 \cdot 3 = 0 \text{ to } \dim H^0(X; \mathcal{O}_X(D)) \\ 2 \cdot 3 = 6 \text{ to } \dim H^1(X; \mathcal{O}_X(D)) \\ 0 \cdot 3 = 0 \text{ to } \dim H^2(X; \mathcal{O}_X(D)) \end{cases}$$

If one computes all other squarefree monomials $\mathcal{Q}(S^k_\alpha)$ arising from unions of SR generators in this fashion, one arrives at the result

$$\dim H^\bullet(X; \mathcal{O}_X(-4H)) = (0,\ 15,\ 0).$$

Note: This procedure is **completely algorithmic**, no specific "tricks" particular to the geometry are required!

Implementation: cohomCalg

You do not have to do this by hand!

$\rightsquigarrow$ high-speed, cross-platform C++ implementation **cohomCalg**

- Windows / Mac / Linux
- open source, GPLv3

Input file for dP_3 example

```
% The vertices and GLSM charges:
vertex x1 | GLSM: ( 1, 0, 0, 1 );
vertex x2 | GLSM: ( 1, 0, 1, 0 );
vertex x3 | GLSM: ( 1, 1, 0, 0 );
vertex x4 | GLSM: ( 0, 1, 0, 0 );
vertex x5 | GLSM: ( 0, 0, 1, 0 );
vertex x6 | GLSM: ( 0, 0, 0, 1 );

% The Stanley-Reisner ideal:
srideal [u1*u2, u1*u3, u1*u6,
u2*u3, u2*u5, u3*u4,
u4*u5, u4*u6, u5*u6];

% And finally the requested
% line bundle cohomologies:
ambientcohom O(-4, 0, 0,0);
```

Implementation: cohomCalg

Command line: `cohomcalg dP3.in > dP3.out`

Output file for dP_3 example

```
      (...preamble...)

Cohomology dimensions:
======================
dim H^i(A; O( -4, 0, 0, 0 )) = ( 0, 15, 0 )
```

The package including a full documentation, source, etc. is available at

http://wwwth.mppmu.mpg.de...
.../members/blumenha/cohomcalg/ or just → Google for **cohomCalg**

Outlook: Consider equivariant cohomology.
$\rightsquigarrow$ "Cohomology of Line Bundles - Applications" *...work in progress...*