A Computational Tool for Line Bundle Cohomology

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- The algorithm
- Example: $dP_3$
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- **Algorithm:** R. Blumenhagen, B. Jurke, T. Rahn, H. Roschy
- **Proof A:** T. Rahn, H. Roschy
- **Proof B:** S.-T. Jow
Motivation: Why line bundle cohomology?

In the context of toric geometry line bundles are appearing all over:

- **Description of the tangent bundle:**

\[
0 \rightarrow \mathcal{O}_X^{\oplus r} \xrightarrow{\alpha} \bigoplus_{i=1}^{n} \mathcal{O}_X(D_i) \xrightarrow{\beta} T_X \rightarrow 0
\]

- **Monad and extension bundle** constructions:

\[
0 \rightarrow V \xrightarrow{f} \bigoplus_{i=1}^{r_B} \mathcal{O}_X(b_i) \xrightarrow{g} \bigoplus_{i=1}^{r_C} \mathcal{O}_X(c_i) \rightarrow 0
\]

\[
0 \rightarrow \bigoplus_{i=1}^{r_A} \mathcal{O}_X(a_i) \xrightarrow{f} W \xrightarrow{g} \bigoplus_{i=1}^{r_C} \mathcal{O}_X(c_i) \rightarrow 0
\]

- **Koszul sequence** for subspaces:

\[
0 \rightarrow \mathcal{O}_X(-D) \xleftarrow{} \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0
\]
Motivation: Why line bundle cohomology?

From those short exact sequences one considers the induced long exact sequences of the sheaf cohomology:

\[ 0 \to \mathcal{O}_X^{\oplus r} \to \bigoplus_{k=1}^n \mathcal{O}_X(D_k) \to T_X \to 0 \]

\[ 0 \to H^0(X; \mathcal{O}_X)^{\oplus r} \to \bigoplus_{k=1}^n H^0(X; \mathcal{O}_X(D_k)) \to H^0(X; T_X) \to H^1(X; \mathcal{O}_X)^{\oplus r} \to \bigoplus_{k=1}^n H^1(X; \mathcal{O}_X(D_k)) \to H^1(X; T_X) \to H^2(X; \mathcal{O}_X)^{\oplus r} \to \bigoplus_{k=1}^n H^2(X; \mathcal{O}_X(D_k)) \to H^2(X; T_X) \to \ldots \]
Description of the algorithm

Ultimately, we are interested in computing \( \dim H^i(X; \mathcal{O}_X(D)) \).

Setting:
- \( X \) toric variety
- homogeneous coordinates
  \( H = \{x_1, \ldots, x_n\} \)
- Stanley-Reisner ideal
  \( \text{SR} = \langle S_1, \ldots, S_N \rangle \)

<table>
<thead>
<tr>
<th>coords</th>
<th>GLSM charges</th>
<th>divisor class</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Take a squarefree monomial \( Q = x_{i_1} \cdots x_{i_k} \) of the coordinates \( H \).

**Basic idea:** Count rational functions of the form

\[
R^Q(x_1, \ldots, x_n) = \frac{T(x_{j_1}, \ldots, x_{j_{n-k}})}{x_{i_1} \cdots x_{i_k} \cdot W(x_{i_1}, \ldots, x_{i_k})} \quad T, W \text{ monomials}
\]

complement coordinates \( H \setminus Q \)
Description of the algorithm

Due to a surprising **vanishing result** it actually suffices to consider a restricted number of rational functions:

\[ N_D(Q) := \dim \{ R^Q : \deg R^Q = D \} \]

**Q:** To which cohomology group \( H^i \) does this number contribute? Need to trace back how often the same union of SR ideal generators arises.

\[ S^k_\alpha := \{ S_{\alpha 1}, \ldots, S_{\alpha k} \} \subset \text{SR} \quad \text{set of } k \text{ SR generators} \]

Let \( Q(S^k_\alpha) \) be the squarefree monomial from the union of those generators.

\[ N(S^k_\alpha) := |Q(S^k_\alpha)| - k \]

Gives a measure for the multiplicity of coordinates in different SR generators, from which the union \( Q(S^k_\alpha) \) arises

(working through the powerset of SR generators)
Description of the algorithm

Note that $N(S^k_{\alpha}) \in \{-N, \ldots, n\}$, where

$$\begin{cases} N \text{ number of SR gens} \\ n \text{ number of coords.} \end{cases}$$

Count the number of combinations of SR generators yielding the same monomial and multiplicity measure:

$$\dim \mathcal{C}^i(Q) := \# \left\{ S^k_{\alpha} \subset \text{SR} : \begin{array}{c} Q(S^k_{\alpha}) = Q \\ N(S^k_{\alpha}) = i \end{array} \right\}$$

Consider then the cohomology $h^i(Q)$ of the sequence

$$0 \rightarrow \mathcal{C}^{-N} \rightarrow \mathcal{C}^{-N+1} \rightarrow \ldots \rightarrow \mathcal{C}^n \rightarrow 0$$

In most cases those multiplicity factors will either be 0 or 1.

(In precise mathematical terms this sequence corresponds to a subcomplex of the full Taylor resolution of the Stanley-Reisner ring $\mathbb{C}[H]/\text{SR}$, which defines the mappings $\mathcal{C}^i \rightarrow \mathcal{C}^{i+1}$.)
Algorithm overview

**Dimension of line bundle sheaf cohomology**

\[
\dim H^i(X; \mathcal{O}_X(D)) = \sum_{Q} \left( \begin{array}{c}
\mathcal{h}^i(Q) \cdot \mathcal{N}_D(Q) \\
\text{multiplicity factor}
\end{array} \right) \\
\text{where the sum ranges over unions of SR generators}
\]

1. Determine all monomials \( Q \) from unions of SR gens.
2. For each such \( Q \) compute the corresponding numbers of SR gen. combinations \( \dim \mathcal{C}^i(Q) \)
3. From those determine the multiplicity factors \( h^i(Q) \)
4. For each \( Q \) where \( h^i(Q) \neq 0 \) count the number of rational functions \( \mathcal{N}_D(Q) \).
5. Sum over all relevant contributions \( h^i(Q) \cdot \mathcal{N}_D(Q) \).

\[ \rightsquigarrow \dim H^i(X; \mathcal{O}_X(D)) \]
Example: del-Pezzo-3 surface

Consider the monomial \( Q = x_1 x_2 x_3 \):

\[
\begin{align*}
\{x_1 x_2, x_1 x_3\} & \quad N = 3 - 2 = 1 \\
\{x_1 x_2, x_2 x_3\} & \quad N = 3 - 2 = 1 \\
\{x_1 x_3, x_2 x_3\} & \quad N = 3 - 2 = 1 \\
\{x_1 x_2, x_1 x_3, x_2 x_3\} & \quad N = 3 - 3 = 0
\end{align*}
\]

Yields the sequence

\[
0 \rightarrow \mathfrak{c}^0 = \mathbb{C} \rightarrow \mathfrak{c}^1 = \mathbb{C}^3 \rightarrow 0
\]

which leads to

\[
\mathcal{h}^1(Q) = \mathcal{h}^1(x_1 x_2 x_3) = 2.
\]

In total one finds:

- 46 monomials \( Q(S^k_\alpha) \) arise from unions of SR gens
- 34 monomials thereof have \( \mathcal{h}^i(Q(S^k_\alpha)) \neq 0 \) for some \( i \)
Example: del-Pezzo-3 surface

For the monomial $Q = x_1 x_2 x_3$ we now have to count rational functions of the form:

$$R^Q(x_1, \ldots, x_6) = \frac{T(x_4, x_5, x_6)}{x_1 x_2 x_3 \cdot W(x_1, x_2, x_3)}, \quad T, W \text{ monomials}$$

Let $\|x_i\| \in \mathbb{N}$ denote the exponent of the corresponding coordinate.

$$\deg R^Q = \left( - \|x_1\| - \|x_2\| - \|x_3\| - 3, \right.$$

$$\|x_4\| - \|x_3\| - 1,$$

$$\|x_5\| - \|x_2\| - 1,$$

$$\|x_6\| - \|x_1\| - 1 \right)$$

For $D = -4H = (-4, 0, 0, 0)$ we find the following solutions:

$$\mathcal{N}_D(Q) = \dim \{ R^Q : \deg R^Q = -4H \}$$

$$= \# \left\{ \frac{x_4 x_5 x_6^2}{x_1 x_2 x_3 \cdot x_1}, \frac{x_4 x_5^2 x_6}{x_1 x_2 x_3 \cdot x_2}, \frac{x_4^2 x_5 x_6}{x_1 x_2 x_3 \cdot x_4} \right\} = 3$$
Example: del-Pezzo-3 surface

Then due to $h^1(Q) = 2$ and $N_D(Q) = 3$ the contribution of the squarefree monomial $Q = x_1x_2x_3$ to the cohomology $H^i(X; \mathcal{O}_X(D))$ is:

$$h^i(Q) \cdot N_D(Q) \sim \begin{cases} 
0 \cdot 3 = 0 & \text{to } \dim H^0(X; \mathcal{O}_X(D)) \\
2 \cdot 3 = 6 & \text{to } \dim H^1(X; \mathcal{O}_X(D)) \\
0 \cdot 3 = 0 & \text{to } \dim H^2(X; \mathcal{O}_X(D))
\end{cases}$$

If one computes all other squarefree monomials $Q(S^k_{\alpha})$ arising from unions of SR generators in this fashion, one arrives at the result

$$\dim H^\bullet(X; \mathcal{O}_X(-4H)) = (0, 15, 0).$$

Note: This procedure is completely algorithmic, no specific “tricks” particular to the geometry are required!
Implementation: cohomCAlg

You do not have to do this by hand!

cohomCAlg

~⇒ high-speed, cross-platform C++ implementation cohomCAlg
- Windows / Mac / Linux
- open source, GPLv3

Input file for $dP_3$ example

```%
% The vertices and GLSM charges:
vertex x1 | GLSM: ( 1, 0, 0, 1 );
vertex x2 | GLSM: ( 1, 0, 1, 0 );
vertex x3 | GLSM: ( 1, 1, 0, 0 );
vertex x4 | GLSM: ( 0, 1, 0, 0 );
vertex x5 | GLSM: ( 0, 0, 1, 0 );
vertex x6 | GLSM: ( 0, 0, 0, 1 );
%
% The Stanley-Reisner ideal:
srideal [u1*u2, u1*u3, u1*u6,
u2*u3, u2*u5, u3*u4,
u4*u5, u4*u6, u5*u6];
%
% And finally the requested
% line bundle cohomologies:
ambientcohom 0(-4, 0, 0,0);
```
Implementation: cohomCalg

Command line: cohomcalg dP3.in > dP3.out

Output file for $dP_3$ example

(...preamble...)

Cohomology dimensions:
======================
dim $H^i(A; 0(-4, 0, 0, 0)) = (0, 15, 0)$

The package including a full documentation, source, etc. is available at

http://wwwth.mppmu.mpg.de... or just → Google for cohomCalg

.../members/blumenha/cohomcalg/

Outlook: Consider equivariant cohomology.

⇝ "Cohomology of Line Bundles - Applications" ...work in progress...