Cohomology of Toric Varieties and Applications

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Contents

- The algorithm
- cohomCalg
- Applications

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with R. Blumenhagen, T. Rahn, H. Roschy

- Algorithm: arXiv:1003.5217
- Applications: arXiv:1010.3717
Motivation: Why line bundle cohomology?

**Pure line bundles**
- Type IIB orientifolds: Abelian fluxes, chiral spectrum
- Type IIB & F-theory: Instanton zero modes, fluxes

**Vector bundles from line bundles**

Heterotic model building: Holomorphic vector bundle $V$ over Calabi-Yau 3-fold $X$ for the breaking of the gauge group.

- Most vector bundles are constructed as monads:

  $0 \longrightarrow V \longleftarrow \bigoplus \mathcal{O}_X(b_i) \longrightarrow \bigoplus \mathcal{O}_X(c_j) \longrightarrow 0$

- The tangent bundle $T_X$ can be described as a monad.
- The vector bundle moduli can be computed from $\text{End}(V) \cong V \otimes V^*$. 
Motivation: Why line bundle cohomology?

From those short exact sequences of sums of line bundles one considers the induced long exact sequences of the sheaf cohomology, e.g.:

\[ 0 \rightarrow \mathcal{O}_X^{\oplus r} \rightarrow \bigoplus_k \mathcal{O}_X(D_k) \rightarrow T_X \rightarrow 0 \]

\[ 0 \rightarrow H^0(X; \mathcal{O}_X)^{\oplus r} \rightarrow \bigoplus_k H^0(X; \mathcal{O}_X(D_k)) \rightarrow H^0(X; T_X) \rightarrow \]

\[ \rightarrow H^1(X; \mathcal{O}_X)^{\oplus r} \rightarrow \bigoplus_k H^1(X; \mathcal{O}_X(D_k)) \rightarrow H^1(X; T_X) \rightarrow \]

\[ \rightarrow H^2(X; \mathcal{O}_X)^{\oplus r} \rightarrow \bigoplus_k H^2(X; \mathcal{O}_X(D_k)) \rightarrow H^2(X; T_X) \rightarrow \ldots \]

Everything boils down to the computation of line bundle cohomology.
Description of the algorithm

Ultimately, we are interested in computing \( \dim H^i(X; \mathcal{O}_X(D)) \).

**Input data: toric variety \( X \)**
- homogeneous coordinates \( H = \{x_1, \ldots, x_n\} \)
- associated GLSM charges \( Q^a_i \) for each \( x_i \)
- Stanley-Reisner ideal \( \text{SR} = \langle S_1, \ldots, S_N \rangle \)

Take a squarefree monomial \( Q = x_{i_1} \cdots x_{i_k} \) of the coordinates \( H \).

Consider monomials of the form

\[
R^Q(x_1, \ldots, x_n) = \frac{T(x_{j_1}, \ldots, x_{j_{n-k}})}{x_{i_1} \cdots x_{i_k} \cdot W(x_{i_1}, \ldots, x_{i_k})} \cdot Q \cdot W(x_{i_1}, \ldots, x_{i_k})^{-1} \cdot x_{i_1}^{-1} \cdots x_{i_k}^{-1}
\]

where \( T, W \) monomials

\[
= x_{j_1}^{\rho_{j_1}} \cdots x_{j_{n-k}}^{\rho_{j_{n-k}}} \cdot x_{i_1}^{-1-\rho_{i_1}} \cdots x_{i_k}^{-1-\rho_{i_k}}
\]
Description of the algorithm

**Step 1: Count number of monomials with degree of $D$**

\[
\mathcal{N}_D(Q) := \dim \left\{ R^Q : \deg_{GLSM} R^Q = D \right\}
\]

$Q = \text{squarefree monomial from the union of the coordinates in SR ideal generators}$

Determine to which cohomology group dimension $h^i(X; \mathcal{O}_X(D))$ the number $\mathcal{N}_D(Q)$ contributes.

$\rightarrow \text{Trace back how often the same } Q \text{ arises.}$

For each $Q$ build up an **abstract simplex** $\Gamma^Q := \{ S \in SR : Q(S) = Q \}$ with $k$-faces

\[
F_k(\Gamma^Q) := \left\{ S \in \Gamma^Q : |S| = k + 1 \right\}.
\]
Description of the algorithm

\[ \phi_k : F_k(\Gamma Q) \longrightarrow F_{k-1}(\Gamma Q) \]

\[ e_\rho \mapsto \sum_{s \in \rho} \operatorname{sign}(s, \rho) e_{\rho-\{s\}} \]

defines the boundary mappings, where \( e_{\rho-\{s\}} = 0 \) if \( e_{\rho-\{s\}} \notin \Gamma Q \).

Step 2: Multiplicity factor and group contribution

Consider the (reduced) homology \( \tilde{H}_i(\Gamma Q) \) and define the multiplicity factors

\[ h_i(Q) := \dim \tilde{H}_{|Q|-i-1}(\Gamma Q) \]

Those multiplicity factors are 0 or 1 in most cases—but not always. "Dirty trick": Via exactness it often suffices to determine just \( \dim F_k(\Gamma Q) \).
Algorithm overview

**Dimension of line bundle sheaf cohomology**

\[
\dim H^i(X; \mathcal{O}_X(D)) = \sum_{Q} \#(\text{suitable monomials } R^Q) \cdot h_i(Q) \cdot N_D(Q)
\]

where the sum ranges over square-free monomials from unions of SR generators.

1. Determine all monomials \( Q \) from unions of SR gens.
2. For each such \( Q \) compute the corresponding numbers of SR gen. combinations \( F_k(\Gamma^Q) \)
3. From those determine the multiplicity factors \( h_i(Q) \)
4. For each \( Q \) where \( h_i(Q) \neq 0 \) count the number of rational functions \( N_D(Q) \).
5. Sum over all relevant contributions \( h_i(Q) \cdot N_D(Q) \).

→ completely algorithmic
From the cohomology of a toric ambient space one can descent to the cohomology of a hypersurface, e.g. a Calabi-Yau hypersurface like $\mathbb{P}^4[5]$.

### The Koszul sequence

Let $S$ be a hypersurface (i.e. divisor) in a toric variety $X$ and $T$ be an arbitrary second divisor of $X$.

$$0 \rightarrow \mathcal{O}_X(T - S) \hookrightarrow \mathcal{O}_X(T) \rightarrow \mathcal{O}_S(T) \rightarrow 0.$$  

- Line bundles on ambient space $X$
- Line bundle on hypersurface $S \subset X$

→ Compute $H^i(S; \mathcal{O}_S(T))$ via induced long exact cohomology sequence.

The complete intersection $S = S_1 \cap \cdots \cap S_t$ of several hypersurfaces $S_i \subset X$ can be handled by iteration.
Application: Finite group actions

The explicit form of the monomials $R^Q$ contributing to $\dim H^i(X; \mathcal{O}_X(D))$ allows to consider finite group actions and the quotient’s cohomology.

First considered for $\mathbb{Z}_2$ orientifolds by Cvetic-García-Etxebarria-Halverson; arXiv:1009.5386

Equivariant structure on line bundles

Let $G$ be a finite group acting holomorphically on $X$. The group element action $g : X \rightarrow X$ on the base space may be lifted to the bundle mapping $\phi_g : L \rightarrow L$. If

$$\phi_g \circ \phi_h = \phi_{gh}$$

this defines an equivariant structure.

$\Rightarrow$ Apply involution on coordinates $x_i$ directly to the monomials $R^Q$.

But: (projectively) equivalent involutions on the base generally define different equivariant structures!
Application: Finite group actions

\(G\)-action induces a splitting of the cohomology

\[ H^i(X; \mathcal{O}_X(D)) = H^i_{\text{inv}}(X; \mathcal{O}_X(D)) \oplus H^i_{\text{non-inv}}(X; \mathcal{O}_X(D)) \]

The dimensions of those splittings can be computed by applying the uplifted \(G\)-action on the monomials counted in \(N_D(Q) = N_{\text{inv}} \oplus N_{\text{non-inv}}\).

**Quotient space cohomology**

\[ h^i(X/G; \widehat{\mathcal{O}_X(D)}) = h^i_{\text{inv}}(X; \mathcal{O}_X(D)) = \sum Q h^i(Q) \cdot N_{D, \text{inv}}(Q) \]

bundle over the quotient space

Ex.: \(\mathcal{O}(-6) \longrightarrow \mathbb{CP}^2\) with \(\mathbb{Z}_3\)-action:

\[ \phi_g : (x_1, x_2, x_3) \mapsto (\alpha x_1, \alpha^2 x_2, x_3) \]

where \(\alpha := \sqrt[3]{1} = e^{\frac{2\pi i}{3}}\)

inv: \(\frac{1}{x_4 x_2 x_3}\), non-inv: \(\frac{\alpha^2}{x_1^3 x_2^2 x_3}\)
Implementation: cohomCalg

cohomCalg

cohomCalg
high-speed, cross-platform C++ implementation cohomCalg
- Windows / Mac / Linux
- open source, GLPv3
- multi-core support

Google for cohomCalg

or try the core algorithm online:

cohomcalg.benjaminjurke.net

cohomCalg Koszul extension
Mathematica interface
- Hypersurfaces & complete intersections
- (co-)tangent bundle, $\Lambda^2 T^* S$
- Hodge diamond
- Monads
cohomCalg: del Pezzo-3 surface

Example: Line bundles on toric variety $dP_3$

The C++ core program takes care of the actual algorithm that computes line bundles on toric spaces.
Example: The resolved $\mathbb{P}_{11222}[8]$

```
In[181]:= P11222BlowUp = {
    (*Coordinates*) {v1, v2, v3, v4, v5, vX},
    (*Stanley Reisner*) {{v1, v2}, {v3, v4, v5, vX}},
    (*Equivalence Relations*) {{1, 0}, {1, 0}, {2, 1}, {2, 1}, {2, 1}, {0, 1}}
};
CalabiYauHyperSurface = {{8, 4}};
CohomologyOf["TangentBundle", P11222BlowUp, CalabiYauHyperSurface, "Calabi-Yau", "Verbose2"]
```

```
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</table>

Calculation of TangentBundle cohomology done:

Total number of line bundles: 42
Newly computed: 18
Total time needed: 0.593 seconds
cohomCalc: Hodge diamond of a CICY 4-fold

Example: CICY 4-fold in the context of F-Theory GUT Vacua

The Mathematica frontend provides convenient functionality, utilizing the previously discussed methods.
Conclusions

Presented material

- An easy and efficient way to compute the line bundle cohomology of a toric variety.

- The algorithm implementation cohomCalg.

- Methods for the computation of various vector bundles on toric subspaces with a convenient Mathematica frontend.

- An extension of the algorithm to compute quotient space cohomology of a toric variety, i.e. methods to calculate line bundle cohomology on orbifolds and orientifolds.

Thank you!