Cohomology of Toric Varieties and Applications

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Contents

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with R. Blumenhagen, T. Rahn, H. Roschy

- **Algorithm:** arXiv:1003.5217
- **Proofs:** arXiv:1006.2392 & S.-T. Jow: arXiv:1006.0780
- **Applications:** arXiv:1010.3717

Motivation: Why line bundle cohomology?

Pure line bundles

- Type IIB orientifolds: Abelian fluxes, chiral spectrum
- Type IIB & F-theory: Instanton zero modes, fluxes

Vector bundles from line bundles

Heterotic model building: Holomorphic vector bundle V over Calabi-Yau 3-fold X for the breaking of the gauge group.

- Most vector bundles are constructed as monads:

$$0 \longrightarrow V \hookrightarrow \bigoplus \mathcal{O}_X(b_i) \twoheadrightarrow \bigoplus \mathcal{O}_X(c_j) \longrightarrow 0$$

- The tangent bundle T_X can be described as a monad.
- The vector bundle moduli can be computed from $\mathrm{End}(V) \cong V \otimes V^*$.

Motivation: Why line bundle cohomology?

From those short exact sequences of sums of line bundles one considers the **induced long exact sequences** of the sheaf cohomology, e.g.:

$$0 \longrightarrow \mathcal{O}_X^{\oplus r} \hookrightarrow \bigoplus_k \mathcal{O}_X(D_k) \twoheadrightarrow T_X \longrightarrow 0$$

$$\Downarrow$$

$$0 \longrightarrow H^0(X;\mathcal{O}_X)^{\oplus r} \longrightarrow \bigoplus_k H^0(X;\mathcal{O}_X(D_k)) \longrightarrow H^0(X;T_X) \longrightarrow$$

$$\longrightarrow H^1(X;\mathcal{O}_X)^{\oplus r} \longrightarrow \bigoplus_k H^1(X;\mathcal{O}_X(D_k)) \longrightarrow H^1(X;T_X) \longrightarrow$$

$$\longrightarrow H^2(X;\mathcal{O}_X)^{\oplus r} \longrightarrow \bigoplus_k H^2(X;\mathcal{O}_X(D_k)) \longrightarrow H^2(X;T_X) \longrightarrow \dots$$

Everything boils down to the computation of line bundle cohomology.

Description of the algorithm

Ultimately, we are interested in ➙ computing $\dim H^i(X; \mathcal{O}_X(D))$.

Input data: toric variety X

- homogeneous coordinates $H = \{x_1, \ldots, x_n\}$
- associated GLSM charges Q_i^a for each x_i
- Stanley-Reisner ideal $\mathrm{SR} = \langle \mathcal{S}_1, \ldots, \mathcal{S}_N \rangle$

Take a squarefree monomial $\mathcal{Q} = x_{i_1} \cdots x_{i_k}$ of the coordinates H.

Consider monomials of the form

$$R^{\mathcal{Q}}(x_1, \ldots, x_n) = \frac{T(\overbrace{x_{j_1}, \ldots, x_{j_{n-k}}}^{\text{complement coordinates } H \setminus \mathcal{Q}})}{\underbrace{x_{i_1} \cdots x_{i_k}}_{\mathcal{Q}} \cdot W(\underbrace{x_{i_1}, \ldots, x_{i_k}}_{\text{coordinates in } \mathcal{Q}})}$$

T, W monomials

$$= x_{j_1}^{\rho_{j_1}} \cdots x_{j_{n-k}}^{\rho_{j_{n-k}}} \cdot x_{i_1}^{-1-\rho_{i_1}} \cdots x_{i_k}^{-1-\rho_{i_k}}$$

Description of the algorithm

Step 1: Count number of monomials with degree of D

$$\boxed{\mathcal{N}_D(\mathcal{Q}) := \dim \left\{ R^{\mathcal{Q}} : \deg_{\mathrm{GLSM}} R^{\mathcal{Q}} = D \right\}}$$

$\mathcal{Q} =$ squarefree monomial from the union of the coordinates in SR ideal generators

Determine to which cohomology group dimension $h^i(X; \mathcal{O}_X(D))$ the number $\mathcal{N}_D(\mathcal{Q})$ contributes.

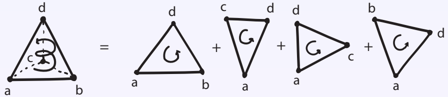
➙ Trace back how often the same $\mathcal{Q}$ arises.

For each $\mathcal{Q}$ build up an abstract simplex $\Gamma^{\mathcal{Q}} := \{ S \subset \mathrm{SR} : \mathcal{Q}(S) = \mathcal{Q} \}$ with k-faces

$$F_k(\Gamma^{\mathcal{Q}}) := \left\{ S \in \Gamma^{\mathcal{Q}} : |S| = k + 1 \right\}.$$

Description of the algorithm

$$\phi_k : F_k(\Gamma^{\mathcal{Q}}) \longrightarrow F_{k-1}(\Gamma^{\mathcal{Q}})$$
$$e_\rho \mapsto \sum_{s\in\rho} \mathrm{sign}(s,\rho) e_{\rho-\{s\}}$$



defines the boundary mappings, where $e_{\rho-\{s\}} = 0$ if $e_{\rho-\{s\}} \notin \Gamma^{\mathcal{Q}}$.

Step 2: Multiplicity factor and group contribution

Consider the (reduced) homology $\tilde{H}_\bullet(\Gamma^{\mathcal{Q}})$ and define the multiplicity factors

$$\mathfrak{h}_i(\mathcal{Q}) := \dim \tilde{H}_{|\mathcal{Q}|-i-1}(\Gamma^{\mathcal{Q}})$$

Those multiplicity factors are 0 or 1 in most cases—but not always.

"Dirty trick": Via exactness it often suffices to determine just $\dim F_k(\Gamma^{\mathcal{Q}})$.

Algorithm overview

Dimension of line bundle sheaf cohomology

$$\dim H^i(X;\mathcal{O}_X(D)) = \sum_{\mathcal{Q}} \underbrace{\mathfrak{h}_i(\mathcal{Q})}_{\text{multiplicity factor}} \cdot \overbrace{\mathcal{N}_D(\mathcal{Q})}^{\#(\text{suitable monomials } R^{\mathcal{Q}})}$$

sum ranges over square-free monomials from unions of SR generators

1. Determine all monomials $\mathcal{Q}$ from unions of SR gens.
2. For each such $\mathcal{Q}$ compute the corresponding numbers of SR gen. combinations $F_k(\Gamma^{\mathcal{Q}})$
3. From those determine the multiplicity factors $\mathfrak{h}_i(\mathcal{Q})$
4. For each $\mathcal{Q}$ where $\mathfrak{h}_i(\mathcal{Q}) \neq 0$ count the number of rational functions $\mathcal{N}_D(\mathcal{Q})$.
5. Sum over all relevant contributions $\mathfrak{h}_i(\mathcal{Q}) \cdot \mathcal{N}_D(\mathcal{Q})$.

→ **completely algorithmic**

Application: Hypersurfaces & the Koszul sequence

From the cohomology of a toric ambient space one can descent to the cohomology of a hypersurface, e.g. a Calabi-Yau hypersurface like $\mathbb{P}^4[5]$.

The Koszul sequence

Let S be a hypersurface (i.e. divisor) in a toric variety X and T be an arbitrary second divisor of X.

$$0 \longrightarrow \underbrace{\mathcal{O}_X(T-S) \hookrightarrow \mathcal{O}_X(T)}_{\text{line bundles on ambient space } X} \twoheadrightarrow \underbrace{\mathcal{O}_S(T)}_{\text{line bundle on hypersurface } S \subset X} \longrightarrow 0.$$

➙ Compute $H^i(S; \mathcal{O}_S(T))$ via induced long exact cohomology sequence.

The complete intersection $S = S_1 \cap \cdots \cap S_t$ of several hypersurfaces $S_i \subset X$ can be handled by iteration.

Application: Finite group actions

The explicit form of the monomials $R^{\mathcal{Q}}$ contributing to $\dim H^i(X; \mathcal{O}_X(D))$ allows to consider finite group actions and the quotient's cohomology.

[First considered for $\mathbb{Z}_2$ orientifolds by Cvetic-García-Etxebarria-Halverson; arXiv:1009.5386]

Equivariant structure on line bundles

Let G be a finite group acting holomorphically on X. The group element action $g : X \longrightarrow X$ on the base space may be lifted to the bundle mapping $\phi_g : L \longrightarrow L$. If

$$\phi_g \circ \phi_h = \phi_{gh}$$

$$\begin{array}{ccc} L & \overset{\phi_g}{\dashrightarrow} & L \\ {\scriptstyle \pi}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\;g\;} & X \end{array}$$

this defines an equivariant structure.

➙ Apply involution on coordinates x_i directly to the monomials $R^{\mathcal{Q}}$.

But: (projectively) equivalent involutions on the base generally define *different* equivariant structures!

G-action induces a splitting of the cohomology

$$H^i(X;\mathcal{O}_X(D)) = H^i_{\text{inv}}(X;\mathcal{O}_X(D)) \oplus H^i_{\text{non-inv}}(X;\mathcal{O}_X(D))$$

The dimensions of those splittings can be computed by applying the uplifted G-action on the monomials counted in $\mathcal{N}_D(\mathcal{Q}) = \mathcal{N}_{\text{inv}} \oplus \mathcal{N}_{\text{non-inv}}$.

Quotient space cohomology

$$h^i(X/G;\underbrace{\widetilde{\mathcal{O}_X(D)}}_{\text{bundle over the quotient space}}) = h^i_{\text{inv}}(X;\mathcal{O}_X(D)) = \sum_{\mathcal{Q}} \mathfrak{h}_i(\mathcal{Q}) \cdot \mathcal{N}_{D,\text{inv}}(\mathcal{Q})$$

Ex.: $\mathcal{O}(-6) \longrightarrow \mathbb{CP}^2$ with $\mathbb{Z}_3$-action:
$\phi_g : (x_1, x_2, x_3) \mapsto (\alpha x_1, \alpha^2 x_2, x_3)$
where $\alpha := \sqrt[3]{1} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$

inv: $\dfrac{1}{x_1^4 x_2 x_3}$, non-inv: $\dfrac{\alpha^2}{x_1^3 x_2^2 x_3}$

Implementation: **cohomCalg**

→ Google for cohomCalg

or try the core algorithm online:

→ `cohomcalg.benjaminjurke.net`

cohomCalg

high-speed, cross-platform
C++ implementation **cohomCalg**

- Windows / Mac / Linux
- open source, GLPv3
- multi-core support

cohomCalg Koszul extension

Mathematica interface

- Hypersurfaces & complete intersections
- (co-)tangent bundle, $\Lambda^2 T^* S$
- Hodge diamond
- Monads

cohomCalg: del Pezzo-3 surface

Example: Line bundles on toric variety dP_3

Open Files

dP3.in

```
% The vertices and GLSM charges:
    vertex u1 | GLSM:  ( 1,  0,  0,  1 );
    vertex u2 | GLSM:  ( 1,  0,  1,  0 );
    vertex u3 | GLSM:  ( 1,  1,  0,  0 );
    vertex u4 | GLSM:  ( 0,  0,  0,  1 );
    vertex u5 | GLSM:  ( 0,  0,  1,  0 );
    vertex u6 | GLSM:  ( 0,  1,  0,  0 );

% The Stanley-Reisner ideal:
    srideal [u1*u2,  u1*u3,  u1*u4,  u2*u3,
             u2*u5,  u3*u6,  u4*u5,  u4*u6,  u5*u6];

% And finally the requested line bundle cohomologies:
    ambientcohom O( -2,  0, -2,  0 );
    ambientcohom O( -3,  2, -2, -1 );
```

Administrator: Command Prompt

```
D:\package\bin>cohomcalg --hideinput --nomonomfile dP3.in

    cohomCalg v0.31
    (compiled on May 25 2011 @ 10:43:46 for Windows x86-64 / 64 bit)
    author: Benjamin Jurke (mail@bjurke.net)
    Koszul extension: Thorsten Rahn (thorsten.rahn@gmail.com)
    Based on the algorithm presented in arXiv:1003.5217

Reading in the input file 'dP3.in'...

Usage and generation of intermediate monomial files deactivated.

Starting computation of secondary sequences...

Computation of secondary cohomologies and contributions complete.
Computation of the target cohomology group dimensions complete.

Cohomology dimensions:
======================
    dim H^i(A; O(  -2,   0,  -2,   0 )) = (   0,   2,   0 )

    dim H^i(A; O(  -3,   2,  -2,  -1 )) = (   0,  10,   0 )


    All done. Programm run successfully completed.
```

The C++ core program takes care of the actual algorithm that computes line bundles on toric spaces.

cohomCalg: Tangent bundle cohomology of hypersurface

Example: The resolved $\mathbb{P}_{11222}[8]$

```
In[181]:= P11222BlowUp = {
      (*Coordinates*){v1, v2, v3, v4, v5, vX},
      (*Stanley Reisner*){{v1, v2}, {v3, v4, v5, vX}},
      (*Equivalence Relations*){{1, 0}, {1, 0}, {2, 1}, {2, 1}, {2, 1}, {0, 1}}
    };
  CalabiYauHyperSurface = {{8, 4}};
  CohomologyOf["TangentBundle", P11222BlowUp, CalabiYauHyperSurface, "Calabi-Yau", "Verbose2"]
```

$\mathcal{O}_S^{r\oplus}$	$\oplus_{k=1}^{n}[\mathcal{O}_S[D_k]]$ $\mathcal{O}_S[1,0]^{\oplus 2} \oplus \mathcal{O}_S[2,1]^{\oplus 3} \oplus \mathcal{O}_S[0,1]$	$\mathcal{E}_S$
2	23	A_{21}
0	3	A_{22}
0	0	A_{23}
2	0	A_{24}
0	0	0

$\Longrightarrow$

$\mathcal{E}_S$
21
3
2
0
0

T_S	$\mathcal{E}_S$	$\oplus_{i=1}^{l}[\mathcal{O}_S[S_i]]$ $\mathcal{O}_S[8,4]$
0	21	104
A_{32}	3	0
A_{33}	2	0
0	0	0
0	0	0

$\Longrightarrow$

T_S
0
86
2
0
0

```
Calculation of TangentBundle cohomology done:
---------------------------------------------
Total number of line bundles:   42
Newly computed:                 18
Total time needed:              0.593 seconds
```

$h^\bullet[T_S]$
0
86
2
0

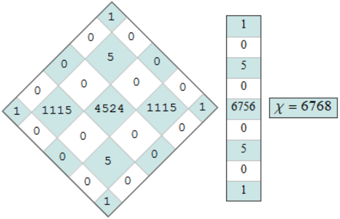
cohomCalg: Hodge diamond of a CICY 4-fold

Example: CICY 4-fold in the context of F-Theory GUT Vacua

```
In[53]:= Example4Fold = {
    (*Coordinates*){v1, v2, v3, v4, v5, v6, v1s, v7, v8, v9, v10},
    (*Stanley Reisner*){{v3, v9}, {v5, v9}, {v7, v10}, {v1, v2, v3}, {v4, v1s, v8}, {v4, v7, v8}, {v4, v8, v9},
                        {v5, v6, v1s}, {v5, v6, v10}, {v1, v2, v6, v1s}},
    (*Equivalence Relations*){{3, 3, 3, 3, 0}, {2, 2, 2, 2, 0}, {1, 0, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, 1, 0},
                        {0, 1, 0, 0, 0}, {0, 1, 1, 0, 0}, {0, 0, 1, 0, 1}, {0, 0, 1, 0, 0}, {0, -1, -1, 1, -1}, {0, 0, 0, 0, 1}}};
CompleteIntersection = {{6, 6, 6, 6, 0}, {0, 0, 2, 1, 1}};

CohomologyOf["HodgeDiamond", Example4Fold, CompleteIntersection, "Calabi-Yau"]
```

```
Calculation of HodgeDiamond cohomology done:
-----------------------------------------------
Total number of line bundles:   478
Newly computed:                 264
Total time needed:              31.233 seconds
```



The Mathematica frontend provides convenient functionality, utilizing the previously discussed methods.

Conclusions

Presented material

- An easy and efficient way to compute the line bundle cohomology of a toric variety.
- The algorithm implementation **cohomCalg**.
- Methods for the computation of various vector bundles on toric subspaces with a convenient Mathematica frontend.
- An extension of the algorithm to compute quotient space cohomology of a toric variety, i.e. methods to calculate line bundle cohomology on orbifolds and orientifolds.

Thank you!