Line bundle valued Cohomology on Toric Varieties

Benjamin Jurke

Northeastern University, Boston MA

AMS 2011 Fall Eastern Sectional Meeting (Cornell)

— Sep 10, 2011 —

Outline

1. Motivation: String Model Building & Phenomenology
2. Description of the Algorithm
3. Computer Implementation

in collaboration with R. Blumenhagen, T. Rahn, H. Roschy
(Max-Planck-Institute für Physik, Munich, Germany)

- **Algorithm:** arXiv:1003.5217
- **Proofs:** arXiv:1006.2392 & S.-T. Jow: arXiv:1006.0780
- **Applications:** arXiv:1010.3717

- **Related:** arXiv:1106.4998 & arXiv:1109.1571

Bundle constructions

In the context of toric geometry line bundles are appearing all over:

- Description of the **tangent bundle**:

$$0 \longrightarrow \mathcal{O}_{\mathcal{X}}^{\oplus r} \xhookrightarrow{\alpha} \bigoplus_{i=1}^{n} \mathcal{O}_{\mathcal{X}}(D_i) \xrightarrow{\beta} T_{\mathcal{X}} \longrightarrow 0$$

- **Monad** and **extension bundle** constructions:

$$0 \longrightarrow \mathcal{V} \xhookrightarrow{f} \bigoplus_{i=1}^{r_B} \mathcal{O}_{\mathcal{X}}(b_i) \xrightarrow{g} \bigoplus_{i=1}^{r_C} \mathcal{O}_{\mathcal{X}}(c_i) \longrightarrow 0$$

$$0 \longrightarrow \bigoplus_{i=1}^{r_A} \mathcal{O}_{\mathcal{X}}(a_i) \xhookrightarrow{f} \mathcal{W} \xrightarrow{g} \bigoplus_{i=1}^{r_C} \mathcal{O}_{\mathcal{X}}(c_i) \longrightarrow 0$$

- **Koszul sequence** for subspaces:

$$0 \longrightarrow \mathcal{O}_{\mathcal{X}}(-D) \hookrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{O}_D \longrightarrow 0$$

1. Motivation: Line bundles and Cohomology

From short exact sequences of sums of line bundles one considers the **induced long exact sequences** of the cohomology, e.g.:

$$0 \longrightarrow \mathcal{O}_{\mathcal{X}}^{\oplus r} \hookrightarrow \bigoplus_k \mathcal{O}_{\mathcal{X}}(D_k) \twoheadrightarrow T_{\mathcal{X}} \longrightarrow 0$$

$$\Downarrow$$

$$\begin{array}{l}
0 \longrightarrow H^0(\mathcal{X};\mathcal{O}_{\mathcal{X}})^{\oplus r} \longrightarrow \bigoplus_k H^0(\mathcal{X};\mathcal{O}_{\mathcal{X}}(D_k)) \longrightarrow H^0(\mathcal{X};T_{\mathcal{X}}) \longrightarrow \\
\longrightarrow H^1(\mathcal{X};\mathcal{O}_{\mathcal{X}})^{\oplus r} \longrightarrow \bigoplus_k H^1(\mathcal{X};\mathcal{O}_{\mathcal{X}}(D_k)) \longrightarrow H^1(\mathcal{X};T_{\mathcal{X}}) \longrightarrow \\
\longrightarrow H^2(\mathcal{X};\mathcal{O}_{\mathcal{X}})^{\oplus r} \longrightarrow \bigoplus_k H^2(\mathcal{X};\mathcal{O}_{\mathcal{X}}(D_k)) \longrightarrow H^2(\mathcal{X};T_{\mathcal{X}}) \longrightarrow \dots
\end{array}$$

1. Motivation: Why line bundle cohomology?

General idea: Avoid evaluating the actual mappings in sequences and try to eliminate/relate as much as possible by exactness considerations.

In the shown examples everything boils down to the computation of line bundle-valued cohomology group dimension $h^i(\mathcal{X}; L_{\mathcal{X}})$.
Known methods:

- **Isomorphisms**: If you can find isomorphisms to spaces with known cohomology, you don't have to compute anything.
- **Spectral sequences**: The method of spectral sequences allows to compute the cohomology of general spaces, but it is extremely laborious to work with.

➙ Find an *algorithmic* method to compute line bundle cohomology!

2. Algorithm: Overview / General Structure

Central idea: Take the Stanley-Reisner ideal I_Σ of a toric variety $\mathcal{X}_\Sigma$ into account.

Resulting formula: (explained on next slides)

$$\boxed{h^i\big(X;\mathcal{O}_{\mathcal{X}}(\alpha)\big) = \sum_{\substack{\sigma\subseteq[n] \\ \tilde{\sigma},\tilde{\tilde{\sigma}}\in\mathcal{P}(I_\Sigma)}} \overbrace{\left|(\alpha,\sigma)\right|}^{\text{elements in ``neg-group''}} \cdot \underbrace{\beta_{|\sigma|-i,\tilde{\sigma}}(S/I_\Sigma)}_{\text{multiplicity factor}}}$$

- Multiplicity factors only depend on geometry of the variety $\mathcal{X}_\Sigma$, not on the bundle $\mathcal{O}_{\mathcal{X}}(\alpha)$ — compute once for all bundles!
- Bundle-specific part is essentially counting the number of integer solutions for a linear system.

Preliminaries

Setting:

- $\mathcal{X}$ smooth toric variety
- Σ toric fan on the vertex set $V \cong [n] := \{1, \ldots, n\}$
- $x_1, \ldots, x_n$ homogeneous coordinates associated to the vertices
- $S := \mathbb{C}[x_1, \ldots, x_n]$ Cox ring of homogeneous coordinates

Define $\hat{\sigma} := [n] \setminus \sigma$ to be the complement to any $\sigma \subseteq [n]$.

Two important ideals in the Cox ring:

- Stanley-Reisner ideal: $I_\Sigma := \langle \boldsymbol{x}^\sigma : \hat{\sigma} \in \Sigma \rangle$
  (generated by products of the coordinates that are not allowed to vanish simultaneously)
- Irrelevant ideal: $B_\Sigma := \langle \boldsymbol{x}^\sigma : \sigma \notin \Sigma \rangle$

Note: I_Σ and B_Σ are *Alexander-dual* to each other.

2. Algorithm: Local cohomology

On a smooth variety Weil divisors are also Cartier and the class group $\mathrm{Cl}(\mathcal{X}) := \mathrm{Div}(\mathcal{X}) / \mathrm{Div}_0(\mathcal{X})$ is equivalent to the Picard group:

$$\mathrm{Cl}(\mathcal{X}) \cong \mathrm{Pic}(\mathcal{X}) \cong \mathbb{Z}^{n-d}$$

The Cox ring can then be decomposed by a $\mathrm{Cl}(\mathcal{X})$-grading:

$$S = \bigoplus_{\alpha \in \mathrm{Cl}(\mathcal{X})} S_\alpha \qquad \text{where} \qquad S_\alpha \cong \Gamma\big(\mathcal{X}; \mathcal{O}_{\mathcal{X}}(\alpha)\big)$$

Relationship between line bundle-valued cohomology and local cohomology:

$$H^i\big(\mathcal{X}; \mathcal{O}_{\mathcal{X}}(\alpha)\big) \cong \underbrace{H^{i+1}_{B_\Sigma}(S)_\alpha}_{\text{localized on irrelevant ideal}} \qquad \text{for } \alpha \in \mathrm{Cl}(\mathcal{X}),\ i \geq 1.$$

[Eisenbud-Mustaţă-Stillman '00]

2. Algorithm: Grading refinement

Each coordinate x_i carries projective weights $Q_i^{(1)}, \dots, Q_i^{(n-d)}$.
Associate a basis vector $\vec{e}_i \in \mathbb{Z}^n$ and define a map

$$\begin{aligned} f : \mathbb{Z}^n &\longrightarrow \mathrm{Cl}(X) \cong \mathbb{Z}^{n-d} \\ \vec{e}_i &\mapsto [D_i] = (Q_i^{(1)}, \dots, Q_i^{(n-d)}), \end{aligned} \quad \text{where} \quad \overbrace{D_i := \{x_i = 0\} \subset \mathcal{X}}^{\text{coordinate divisor}}$$

Use this map to refine the grading on the localized cohomology:

$$H^i(\mathcal{X}; \mathcal{O}_{\mathcal{X}}(\alpha)) = \bigoplus_{\substack{\vec{u} \in \mathbb{Z}^n: \\ f(\vec{u}) = \alpha}} H^{i+1}_{B_\Sigma}(S)_{\vec{u}} \qquad \text{for } \alpha \in \mathrm{Cl}(\mathcal{X}),\ i \geq 1.$$

However, there is a huge redundancy between the $\mathbb{Z}^n$-graded local cohomology groups.

2. Algorithm: Grading redundancy

However, there is a huge redundancy between the $\mathbb{Z}^n$-graded local cohomology groups. Define

$$\operatorname{neg}(\vec{u}) := \{k \in [n] : u_k < 0\}$$

(component indices where $\vec{u}$ has a negative value)

It follows that

$$H^i_{B_\Sigma}(S)_{\vec{u}} \cong H^i_{B_\Sigma}(S)_{\vec{v}} \iff \operatorname{neg}(\vec{u}) = \operatorname{neg}(\vec{v})$$

[Jow '10]

When this $\mathbb{Z}^n$-grading is translated back to monomials, the only distinguishig factor is therefore which coordinates can appear with negative powers.

$$S = \mathbb{C}[x_1, \ldots, x_n] \quad \rightsquigarrow \quad S\left[\frac{1}{\boldsymbol{x}^{\operatorname{neg}(\vec{u})}}\right] = \mathbb{C}\left[x_1, \ldots, x_n, \frac{1}{\boldsymbol{x}^{\operatorname{neg}(\vec{u})}}\right]$$

2. Algorithm: Grouped dimension formula

If one is only interested in the dimension of the cohomology group, this allows for big simplification:

$$h^i\big(\mathcal{X};\mathcal{O}_{\mathcal{X}}(\alpha)\big) = \sum_{\sigma\subseteq[n]} \overbrace{|(\alpha,\sigma)|}^{\text{elements in ``neg-group''}} \cdot \underbrace{h^{i+1}_{B_\Sigma}(S)_\sigma}_{\text{multiplicity factors}}$$

where $\quad (\alpha,\sigma) := \{\vec{u}\in\mathbb{Z}^n : f(\vec{u}) = \alpha,\ \mathrm{neg}(\vec{u}) = \sigma\}.$

Notes:

- The sum reduces to 2^n terms.
- The multiplicity factors can be expressed as the graded Betti number of a free resolution of the Stanley-Reisner ring S/I_Σ:

$$h^{i+1}_{B_\Sigma}(S)_\sigma = \beta_{|\sigma|-i,\tilde{\sigma}}(S/I_\Sigma)$$

Stanley-Reisner ideal reductions

Let $I_\Sigma = \langle S_1, \ldots, S_t \rangle$ be the Stanley-Reisner ideal with t squarefree monomial generators in $x_1, \ldots, x_n$. For any subset $\tau \subseteq [t]$ define

$$\vec{a}_\tau := \deg_{\boldsymbol{x}} \Big(\overbrace{\mathrm{lcm}_{\boldsymbol{x}}\{S_i : i \in \tau\}}^{\text{union of } \boldsymbol{x}\text{-coordinates}} \Big) \in \mathbb{Z}^n$$

$$\mathcal{P}(I_\Sigma) := \{\vec{a}_\tau : \tau \subseteq [t]\}$$

($\vec{a}_\tau$ corresponds to the union of the x_i-coordinates in the generators $S_{\tau_1}, \ldots, S_{\tau_{|\tau|}}$ in terms of the $\mathbb{Z}^n$-grading, $\mathcal{P}(I_\Sigma)$ is the set of all such unions.)

For $\sigma \subseteq [n]$ define $\boldsymbol{x}^\sigma := \prod_{i\in\sigma} x_i$ and $\tilde{\sigma} := \deg_{\boldsymbol{x}}(\boldsymbol{x}^\sigma) \in \mathbb{Z}^n$.

($\vec{a}_\tau$ and $\tilde{\sigma}$ are vectors of 0s and 1s in $\mathbb{Z}^n$ indicating if the x_i is in the monomial)

2. Algorithm: Stanley-Reisner ideal reductions

Main vanishing theorem: Only unions of coordinates in the Stanley-Reisner ideal generators contribute to the dimension:

$$\boxed{h^{i+1}_{B_\Sigma}(S)_\sigma = \beta_{r,\tilde{\sigma}}(S/I_\Sigma) = 0 \quad \text{for all} \quad \sigma \subseteq [n] \text{ where } \tilde{\sigma} \notin \mathcal{P}(I_\Sigma).}$$

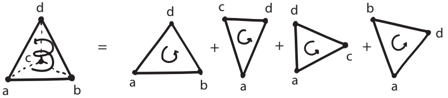
The same result holds for the complement $\hat{\sigma}$, leading to

$$h^i\big(X; \mathcal{O}_X(\alpha)\big) = \sum_{\substack{\sigma \subseteq [n] \\ \tilde{\sigma}, \tilde{\hat{\sigma}} \in \mathcal{P}(I_\Sigma)}} \overbrace{\left|(\alpha, \sigma)\right|}^{\text{elements in ``neg-group''}} \cdot \underbrace{\beta_{|\sigma|-i,\tilde{\sigma}}(S/I_\Sigma)}_{\text{multiplicity factor}}$$

2. Algorithm: Computing multiplicity factors

Let $\sigma \subseteq [n]$ and define an abstract simplicial subcomplex of the full complex $\Delta_{[t]}$ on the Stanley-Reisner ideal generator by

$$\Gamma^{\sigma} := \{\tau \subseteq [t] : \vec{a}_{\tau} = \tilde{\sigma}\}$$



The complex mappings are the usual "ordered alternating sum of faces"-type of standard simplicial complexes.

Main computational result: The multiplicity factors can be computed from the reduced cohomology of Γ^{σ}:

$$\boxed{\beta_{r,\tilde{\sigma}}(S/I_{\Sigma}) = \dim_{\mathbb{C}} \tilde{H}_{r-1}(\Gamma^{\sigma})}$$

3. Implementation: Computational complexity

In practice one can determine the simplicial complexes Γ^{σ} to compute the multiplicity factors and the "selection principle" of $\tilde{\sigma}, \tilde{\hat{\sigma}} \in \mathcal{P}(I_{\Sigma})$ in one step by working through the powerset of Stanley-Reisner ideal generators.

The exponential growth 2^n with the number of vertizes is replaces by an exponential growth 2^t with the number of Stanley-Reisner ideal generators.

Bottom line: Algorithm very efficient for t small!

3. Implementation: **cohomCalg**

→ Google for **cohomCalg**

The core algorithm can be tried online.

cohomCalg

high-speed, cross-platform C++ implementation **cohomCalg**

- Windows / Mac / Linux
- open source, GLPv3
- multi-core support

cohomCalg Koszul extension

Mathematica interface

- Hypersurfaces & complete intersections
- (co-)tangent bundle, $\Lambda^2 T^* S$
- Hodge diamond
- Monads

3. Implementation: **cohomCalg**

Example: Line bundles on toric variety dP_3

Open Files

dP3.in

```
% The vertices and GLSM charges:
    vertex u1 | GLSM:  ( 1,  0,  0,  1 );
    vertex u2 | GLSM:  ( 1,  0,  1,  0 );
    vertex u3 | GLSM:  ( 1,  1,  0,  0 );
    vertex u4 | GLSM:  ( 0,  0,  0,  1 );
    vertex u5 | GLSM:  ( 0,  0,  1,  0 );
    vertex u6 | GLSM:  ( 0,  1,  0,  0 );

% The Stanley-Reisner ideal:
    srideal [u1*u2,  u1*u3,  u1*u4,  u2*u3,
             u2*u5,  u3*u6,  u4*u5,  u4*u6,  u5*u6];

% And finally the requested line bundle cohomologies:
    ambientcohom O( -2,  0, -2,  0 );
    ambientcohom O( -3,  2, -2, -1 );
```

Administrator: Command Prompt

```
D:\package\bin>cohomcalg --hideinput --nomonomfile dP3.in

    cohomCalg v0.31
    (compiled on May 25 2011 @ 10:43:46 for Windows x86-64 / 64 bit)
    author: Benjamin Jurke (mail@bjurke.net)
    Koszul extension: Thorsten Rahn (thorsten.rahn@gmail.com)
    Based on the algorithm presented in arXiv:1003.5217

Reading in the input file 'dP3.in'...


Usage and generation of intermediate monomial files deactivated.

Starting computation of secondary sequences...

Computation of secondary cohomologies and contributions complete.
Computation of the target cohomology group dimensions complete.

Cohomology dimensions:
======================
    dim H^i(A; O(  -2,   0,  -2,   0 )) = (   0,   2,   0 )

    dim H^i(A; O(  -3,   2,  -2,  -1 )) = (   0,  10,   0 )


    All done. Programm run successfully completed.
```

The **C++ core program** takes care of the actual algorithm that computes line bundles on toric spaces.

3. Implementation: **cohomCalg**

Example: The resolved $\mathbb{P}_{11222}[8]$

```
In[181]:= P11222BlowUp = {
      (*Coordinates*){v1, v2, v3, v4, v5, vX},
      (*Stanley Reisner*){{v1, v2}, {v3, v4, v5, vX}},
      (*Equivalence Relations*){{1, 0}, {1, 0}, {2, 1}, {2, 1}, {2, 1}, {0, 1}}
    };
  CalabiYauHyperSurface = {{8, 4}};
  CohomologyOf["TangentBundle", P11222BlowUp, CalabiYauHyperSurface, "Calabi-Yau", "Verbose2"]
```

$\mathcal{O}_S^{r\oplus}$	$\oplus_{k=1}^{n}[\mathcal{O}_S[D_k]]$ $\mathcal{O}_S[1,0]^{\oplus 2} \oplus \mathcal{O}_S[2,1]^{\oplus 3} \oplus \mathcal{O}_S[0,1]$	$\mathcal{E}_S$
2	23	A_{21}
0	3	A_{22}
0	0	A_{23}
2	0	A_{24}
0	0	0

$\Longrightarrow$

$\mathcal{E}_S$
21
3
2
0
0

T_S	$\mathcal{E}_S$	$\oplus_{i=1}^{1}[\mathcal{O}_S[S_i]]$ $\mathcal{O}_S[8,4]$
0	21	104
A_{32}	3	0
A_{33}	2	0
0	0	0
0	0	0

$\Longrightarrow$

T_S
0
86
2
0
0

```
Calculation of TangentBundle cohomology done:
---------------------------------------------
Total number of line bundles:  42
Newly computed:                18
Total time needed:             0.593 seconds
```

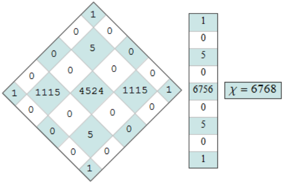
$h^\bullet[T_S]$
0
86
2
0

CICY 4-fold in the context of F-Theory GUT Vacua

```
In[53]:= Example4Fold = {
    (*Coordinates*){v1, v2, v3, v4, v5, v6, v1s, v7, v8, v9, v10},
    (*Stanley Reisner*){{v3, v9}, {v5, v9}, {v7, v10}, {v1, v2, v3}, {v4, v1s, v8}, {v4, v7, v8}, {v4, v8, v9},
                        {v5, v6, v1s}, {v5, v6, v10}, {v1, v2, v6, v1s}},
    (*Equivalence Relations*){{3, 3, 3, 3, 0}, {2, 2, 2, 2, 0}, {1, 0, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, 1, 0},
                              {0, 1, 0, 0, 0}, {0, 1, 1, 0, 0}, {0, 0, 1, 0, 1}, {0, 0, 1, 0, 0}, {0, -1, -1, 1, -1}, {0, 0, 0, 0, 1}}};
CompleteIntersection = {{6, 6, 6, 6, 0}, {0, 0, 2, 1, 1}};

CohomologyOf["HodgeDiamond", Example4Fold, CompleteIntersection, "Calabi-Yau"]
```

```
Calculation of HodgeDiamond cohomology done:
-----------------------------------------------
Total number of line bundles:  478
Newly computed:                264
Total time needed:             31.233 seconds
```



[Grimm-Krause-Weigand '09]

The **Mathematica frontend** provides convenient functionality, utilizing the previously discussed methods.

Conclusions

Presented material:

- An efficient way to compute line bundle-valued cohomology group dimensions of a toric variety.
- The algorithm implementation **cohomCalg**.

Not shown:

- Generalization to discrete group actions on $\mathcal{X}$

Algorithm efficiency made a large scale scan of heterotic $(0, 2)$-models possible, see `arXiv:1106.4998`.