Scanning for SWISS CHEESE geometries

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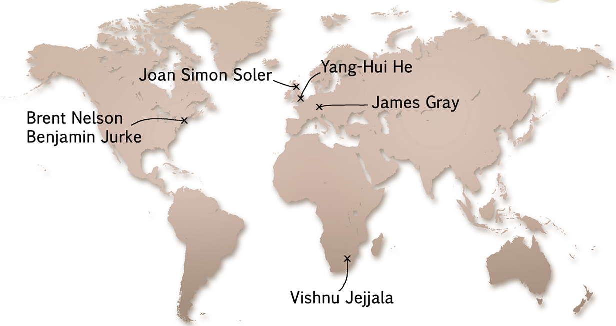
Outline

1. The Large Volume Scenario
2. Swiss Cheese reformulated
3. Implementation

Collaboration:

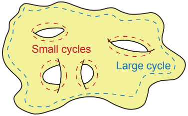


The Large Volume Scenario Claim

$\mathcal{X}$ Calabi-Yau threefold, $D_1, \ldots, D_n \subset \mathcal{X}$ divisors, $\tau_i := \mathrm{vol}(D_i)$

Large Volume Scenario Claim: Let the limit be taken as

$$\text{LV Limit:} \quad \begin{cases} \tau_1, \ldots, \tau_{N_{\mathrm{small}}} \text{ remain small} \\ \mathcal{V} \to \infty \text{ for } \tau_{N_{\mathrm{small}}+1}, \ldots, \tau_{h^{1,1}(\mathcal{X})} \to \infty \end{cases}$$



such that the Kähler potential K and the superpotential W in type IIB $\mathcal{N}=1$ 4d SUGRA

$$K = \langle K_{\mathrm{cs}} \rangle - 2 \ln \left(\hat{\mathcal{V}} + \hat{\xi} \right), \quad W = \langle W_{\mathrm{GVW}} \rangle + \sum_{i=1}^{N_{\mathrm{small}}} A_i(S, U_j) \mathrm{e}^{a_i T_i}$$

Then the scalar potential V admits a set of AdS non-SUSY minima.

[Cicoli-Conlon-Quevedo 2008]

Step 0: What do we have?

Formulation in the LVS Claim:

LV Limit: $\begin{cases} \tau_1, \ldots, \tau_{N_{\text{small}}} \text{ remain small} \\ \mathcal{V} \to \infty \text{ for } \tau_{N_{\text{small}}+1}, \ldots, \tau_{h^{1,1}(\mathcal{X})} \to \infty \end{cases}$

$\Downarrow$

Implied starting assumption: We are in a *"convenient"* basis!
...means that the divisor basis can be directly split up into

$$D_1, \ldots, D_n = \underbrace{D_{\mathrm{L}_1}, \ldots, D_{\mathrm{L}_{n_\mathrm{L}}}}_{\text{large cycle divisors}}, \underbrace{D_{\mathrm{s}_1}, \ldots, D_{\mathrm{s}_{n_\mathrm{s}}}}_{\text{small cycle divisors}}$$

➙ Computationally, that's the most oversimplifying assumption ever!
"Manifest Swiss Cheese"

Step 1: Reformulate Large Volume Limit

$$\text{4-cycle volumes } \tau_i \xleftrightarrow{\text{Poincaré}} \text{Kähler parameters } t^i$$

Central idea: Rewrite LVL in terms of Kähler parameters t^i!

intersection form matrix on divisor D_i: $\quad (\kappa_{(i)})_{jk} := \kappa_{ijk}$

divisor 4-cycle volume: $\quad \tau_i = \frac{1}{2}\kappa_{ijk}t^j t^k = \frac{1}{2}\vec{t}^{*}\kappa_{(i)}\vec{t}$

split Kähler parameter vector: $\quad \vec{t} = \underbrace{\sum_{A=1}^{n_{\mathrm{L}}} \lambda_A \vec{t}_{\mathrm{L}_A}}_{\text{large}} + \underbrace{\sum_{a=1}^{n_{\mathrm{s}}} \gamma_a \vec{t}_{\mathrm{s}_a}}_{\text{small}} \in \mathcal{K}$

$\Downarrow$

In the LVL claim $\mathcal{V} \to \infty$ where $\tau_{N_{\mathrm{small}}+1}, \ldots, \tau_{h^{1,1}(\mathcal{X})} \to \infty$ then corresponds to $\lambda_A \to \infty$.

Step 1: Reformulate Large Volume Limit

What does small and large mean in terms of the Kähler parameters?

$$\tau_i = \frac{1}{2}\Big[\overbrace{\lambda_A\lambda_B \cdot (\vec{t}^{\,*}_{\mathrm{L}_A}\kappa_{(i)}\vec{t}_{\mathrm{L}_B}) + 2\lambda_A\gamma_b \cdot (\vec{t}^{\,*}_{\mathrm{L}_A}\kappa_{(i)}\vec{t}_{\mathrm{s}_b})}^{\text{terms involving large parameters } \lambda_A} + \gamma_a\gamma_b \cdot (\vec{t}^{\,*}_{\mathrm{s}_a}\kappa_{(i)}\vec{t}_{\mathrm{s}_b})\Big]$$

$$\Downarrow$$

$$\begin{array}{ll} \text{large 4-cycles } \tau_I\text{:} & \vec{t}^{\,*}_{\mathrm{L}_A}\kappa_{(I)}\vec{t}_{\mathrm{L}_B} \neq 0 \quad \text{OR} \quad \vec{t}^{\,*}_{\mathrm{L}_A}\kappa_{(I)}\vec{t}_{\mathrm{s}_b} \neq 0 \\ \text{small 4-cycles } \tau_\alpha\text{:} & \underbrace{\vec{t}^{\,*}_{\mathrm{L}_A}\kappa_{(\alpha)}\vec{t}_{\mathrm{L}_B} = 0 \quad \text{AND} \quad \vec{t}^{\,*}_{\mathrm{L}_A}\kappa_{(\alpha)}\vec{t}_{\mathrm{s}_b} = 0}_{\kappa_{(\alpha)}\vec{t}_{\mathrm{L}_A} = 0} \end{array}$$

Step 1: Reformulate Large Volume Limit

Blowup mode condition of the inverse Kähler metric: $K^{-1}_{\alpha\alpha} \sim \mathcal{V}\sqrt{\tau_\alpha}$
For a general Calabi-Yau manifold there is a K^{-1}_{ij} expansion

$$\begin{aligned} K^{-1}_{ij} &= -\frac{2}{9}\left(2\mathcal{V} + \hat{\xi}\right)\kappa_{ijk}t^k + \frac{4\mathcal{V} - \hat{\xi}}{\mathcal{V} - \hat{\xi}}\tau_i\tau_j \\ &= -\frac{4}{9}\mathcal{V}\kappa_{ijk}t^k + 4\tau_i\tau_j + \mathcal{O}(\tfrac{1}{\mathcal{V}^1}) \end{aligned}$$

$$\downarrow$$

$$\frac{K^{-1}_{\alpha\alpha}}{\mathcal{V}} \approx -\frac{4}{9}\kappa_{\alpha\alpha i}t^i = -\frac{4}{9}(\kappa_{(\alpha)}\vec{t})_\alpha \sim \sqrt{\tau_\alpha} = \sqrt{\vec{t}^*\kappa_{(\alpha)}\vec{t}}$$

$\kappa_{(\alpha)}\vec{t}_{\mathrm{L}_A} = 0$ ➙ RHS only depends on small cycle volumes $\vec{t}_{\mathrm{s}_\alpha}$.

➙ requires $(\kappa_{(\alpha)}\vec{t}_{\mathrm{s}_a})_\alpha \neq 0$ for at least one a

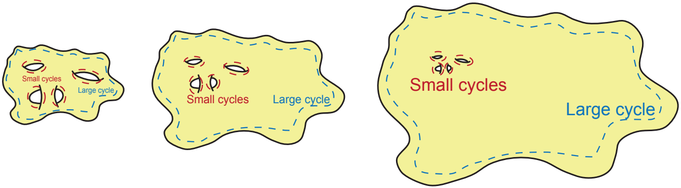
Step 1: Reformulate Large Volume Limit

Furthermore, the $\vec{t}_{\mathrm{L}_A}$ and $\vec{t}_{\mathrm{s}_a}$ have to be a basis (due to Poincaré)

$$\det\left(\vec{t}_{\mathrm{L}_1},\ldots,\vec{t}_{\mathrm{L}_{N_{\text{large}}}},\vec{t}_{\mathrm{s}_1},\ldots,\vec{t}_{\mathrm{s}_{N_{\text{small}}}}\right)\neq 0,$$

which automatically takes care of the large cycles.
Also $\vec{t}$ has to be in the Kähler cone $\mathcal{K}$

$$\mathcal{K}^{\rho}{}_{i}\big(\lambda_A(\vec{t}_{\mathrm{L}_A})^i+\gamma_a(\vec{t}_{\mathrm{s}_a})^i\big)>0$$



Step 1: Reformulated Large Volume Limit

With respect to a "**convenient**" basis we ultimately need to test if

$$\begin{cases} \text{small cycles:} & \kappa_{(\alpha)} \vec{t}_{\mathrm{L}_A} = 0 \\ K^{-1}_{\alpha\alpha} \text{ condition:} & (\kappa_{(\alpha)} \vec{t}_{\mathrm{s}_a})_\alpha \neq 0 \\ \text{non-triviality:} & \det(\vec{t}_{\mathrm{L}_1}, \ldots, \vec{t}_{\mathrm{L}_{N_{\mathrm{large}}}}, \vec{t}_{\mathrm{s}_1}, \ldots, \vec{t}_{\mathrm{s}_{N_{\mathrm{small}}}}) \neq 0 \\ \text{Kähler cone:} & \mathcal{K}^\rho{}_i \big(\lambda_A (\vec{t}_{\mathrm{L}_A})^i + \gamma_a (\vec{t}_{\mathrm{s}_a})^i\big) > 0 \end{cases}$$

has a solution, solving for all $\vec{t}_{\mathrm{L}_A}$, $\vec{t}_{\mathrm{s}_a}$, λ_A and γ_a.

Note that in this equation system only the $\kappa_{(i)}$s are coordinate-dependant.

Step 2: Arbitrary basis considerations

Life is hard... and most bases are rather inconvenient...

Let $\tilde{D}_{\tilde{1}}, \ldots, \tilde{D}_{\tilde{n}}$ be a **generic basis** and $A_i{}^{\tilde{\jmath}}$ a base change matrix relating to the convenient basis $D_1, \ldots, D_n$.

$$
\begin{cases}
\text{small cycles:} & A_\alpha{}^{\tilde{\jmath}} \kappa_{(\tilde{\jmath})} \vec{t}_{\mathrm{L}_A} = 0 \\
K^{-1}_{\alpha\alpha} \text{ condition:} & A_\alpha{}^{\tilde{\imath}} A_\alpha{}^{\tilde{\jmath}} (\kappa_{(\tilde{\imath})} \vec{t}_{\mathrm{s}_a})_{\tilde{\jmath}} \neq 0 \\
\text{non-triviality:} & \det(\vec{t}_{\mathrm{L}_1}, \ldots, \vec{t}_{\mathrm{L}_{N_{\mathrm{large}}}}, \vec{t}_{\mathrm{s}_1}, \ldots, \vec{t}_{\mathrm{s}_{N_{\mathrm{small}}}}) \neq 0 \\
\text{Kähler cone:} & \text{in a moment...} \\
\text{base change:} & \det(A) \neq 0
\end{cases}
$$

Step -1: Kähler cone normalization

general Kähler cone condition: $\displaystyle\sum_{i=1}^{h^{1,1}} \mathcal{K}^{\kappa}{}_{i} t^{i} \geq 0$ for $\kappa = 1, \ldots, n_F$

Kähler cone **simplicial** ➙ $n_F = h^{1,1}$ ➙ $\mathcal{K}$ invertible matrix

Transform intersection form and Kähler cone to standard form via

$$
\begin{aligned}
\hat{D}_{\hat{\imath}} &= \sum_{i} \hat{D}_{i} (\mathcal{K}^{-1})^{i}{}_{\hat{\imath}}, \\
\hat{t}^{\hat{\imath}} &= \sum_{i} \mathcal{K}^{\hat{\imath}}{}_{i} t^{i}, \\
\hat{\kappa}_{\hat{\imath}\hat{\jmath}\hat{k}} &= \sum_{i,j,k} \kappa_{ijk} (\mathcal{K}^{-1})^{i}{}_{\hat{\imath}} (\mathcal{K}^{-1})^{j}{}_{\hat{\jmath}} (\hat{\mathcal{K}}^{-1})^{k}{}_{\hat{k}}.
\end{aligned}
$$

➙ normalized Kähler cone: $\hat{t}^{\hat{\imath}} > 0$

Step 3: Final Swiss Cheese detector

Let $\tilde{D}_{\tilde{1}}, \ldots, \tilde{D}_{\tilde{n}}$ be a **generic basis** with a **normalized Kähler cone**.

$$\begin{cases} \text{small cycles:} & {A_\alpha}^{\tilde{j}} \kappa_{(\tilde{j})} \vec{t}_{\mathrm{L}_A} = 0 \\ K^{-1}_{\alpha\alpha} \text{ condition:} & {A_\alpha}^{\tilde{\imath}} {A_\alpha}^{\tilde{j}} (\kappa_{(\tilde{\imath})} \vec{t}_{\mathrm{s}_a})_{\tilde{j}} \neq 0 \\ \text{non-triviality:} & \det(\vec{t}_{\mathrm{L}_1}, \ldots, \vec{t}_{\mathrm{L}_{N_{\mathrm{large}}}}, \vec{t}_{\mathrm{s}_1}, \ldots, \vec{t}_{\mathrm{s}_{N_{\mathrm{small}}}}) \neq 0 \\ \text{Kähler cone:} & \lambda_A (\vec{t}_{\mathrm{L}_A})^{\tilde{\imath}} + \gamma_a (\vec{t}_{\mathrm{s}_a})^{\tilde{\imath}} > 0 \\ \text{base change:} & \det(A) \neq 0 \end{cases}$$

$$\Downarrow$$

Check the solvability of the system for A, $\vec{t}_{\mathrm{L}_A}$, $\vec{t}_{\mathrm{s}_a}$, λ_A and γ_a over $\mathbb{R}$.

→ Still a demanding task — $\boldsymbol{2(h^{1,1})^2 + h^{1,1}}$ **variables** — but doable!

Redundancy fixings

- $h^{1,1} = 2$: just implement the equation system in Mathematica and brute-force `FindInstance` it.
- $h^{1,1} \geq 3$: the inequality solver in Mathematica is way too slow.

$$\Downarrow$$

Using redundancies in the variables, turn inequalities into equalities:

$$\begin{cases} \text{small cycles:} & A_\alpha{}^{\tilde{\jmath}} \kappa_{(\tilde{\jmath})} \vec{t}_{\mathrm{L}_A} = 0 \\ K^{-1}_{\alpha\alpha} \text{ condition:} & A_\alpha{}^{\tilde{\imath}} A_\alpha{}^{\tilde{\jmath}} (\kappa_{(\tilde{\imath})} \vec{t}_{\mathrm{s}_a})_{\tilde{\jmath}} \neq 0 \\ \text{non-triviality:} & \det(\vec{t}_{\mathrm{L}_1}, \ldots, \vec{t}_{\mathrm{L}_{N_{\mathrm{large}}}}, \vec{t}_{\mathrm{s}_1}, \ldots, \vec{t}_{\mathrm{s}_{N_{\mathrm{small}}}}) = \pm 1 \\ \text{Kähler cone:} & \lambda_A (\vec{t}_{\mathrm{L}_A})^{\tilde{\imath}} + \gamma_a (\vec{t}_{\mathrm{s}_a})^{\tilde{\imath}} > 0 \\ \text{base change:} & \det(A) = \begin{cases} 1 & \text{for } h^{1,1} \text{ even} \\ \pm 1 & \text{for } h^{1,1} \text{ odd} \end{cases} \end{cases}$$

Actual implementation & application

- The $K^{-1}_{\alpha\alpha}$ condition is very non-restrictive and almost never rules out a model for $h^{1,1} > 2$ ➙ ignore it at first.
- In the end Mathematica is simply too slow. Along comes Singular!

➙ SINGULAR

- Using further redundancies, numerous components of the matrix A and the vectors $\vec{t}_{\mathrm{L}_A}$, $\vec{t}_{\mathrm{s}_a}$ can be fixed, reducing the number of variables.

↓

Identifying the right combination of "trickery" was the main effort...

Actual implementation & application

Final implementation of the **Swiss Cheese test:**

1. Fix number of large / small cycles to test.
2. Compute dimension of Gröbner basis of the equation system

$$
\begin{cases}
\text{small cycles:} & A_\alpha{}^{\tilde{\jmath}} \kappa_{(\tilde{\jmath})} \vec{t}_{\mathrm{L}_A} = 0 \\
\text{non-triviality:} & \det(\vec{t}_{\mathrm{L}_1}, \ldots, \vec{t}_{\mathrm{L}_{N_\mathrm{large}}}, \vec{t}_{\mathrm{s}_1}, \ldots, \vec{t}_{\mathrm{s}_{N_\mathrm{small}}}) = \pm 1 \\
\text{base change:} & \det(A) = \begin{cases} 1 & \text{for } h^{1,1} \text{ even} \\ \pm 1 & \text{for } h^{1,1} \text{ odd} \end{cases}
\end{cases}
$$

3. If non-negative, perform primary decomposition of Gröbner basis
4. Add Kähler cone condition and attempt to find solution over $\mathbb{R}$ using Mathematica in at least one component.
5. Check $K^{-1}_{\alpha\alpha}$ condition from the result. ➞ **Swiss Cheese**

Outlook

So far & currently:

- Scanned $h^{1,1} \leq 4$ for $n_{\text{large}} = 1$,
  but test generalizes easily for $n_{\text{large}} > 1$
- Currently scanning $h^{1,1} = 5, 6, 7, 8$
- Also looking at strong vs. weak Swiss Cheese detection
- CICYs for $h^{1,1} \leq 4$ are not Swiss Cheese, but may be for $h^{1,1} \geq 5$

$$\Downarrow$$

Still a lot to discover in the Swiss Cheese landscape...

...the non-toric Swiss Cheese Landscape