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Dimensional Reduction of Spin(7)-Instantons

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Abstract. (Anti-)self-dual instantons are only defined for gauge theories on four-dimensional base spaces. The notion of a gauge instanton can be generalized to dimensions greater than four in the presence of additional geometric structure like special holonomy. Conversely, one can consider the dimensional reduction of such higher-dimensional instanton equations. In the particular case of an 8-dimensional Riemannian manifold M with Spin(7)-holonomy, the instanton equations require the vanishing of the curvature component contained in a 7-dimensional subbundle of $\Lambda^2 \mathrm{T}^* M$. After a short review of the generalization scheme for instanton equations, following the original work of Reyés-Carrion, the dimensional reduction of Spin(7)-instanton equations is carried out in two instances: First, on the product space $Z \times \mathbb{R}$ with Z being a G_2-manifold connections in temporal gauge are considered. Second, for a K3 surface X the translation-invariant connections on $X \times \mathbb{R}^4$ are investigated, which leads to a set of equations similar to the Seiberg-Witten equations.

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Acknowledgement

How I actually finished this second diploma thesis remains a sort of mystery to me and is mostly due to the continual support of my mother and friends, who encouraged me to conclude the work, again and again. However, I have to admit that I'm not thoroughly satisfied with its final state, as it was completed in a great rush due to a strict deadline to the beginning of my physics' Ph.D. position. On the other hand, there always remains something to fix and aspects that should have been covered in greater detail—a lesson learned in the aftermath of my first diploma in physics. Regardless, despite the stress felt in the prospect of a closing deadline, I honestly value the great variety of mathematical subjects learned within that time of preparation, which will undoubtedly prove to be useful in my further work.

Not much has changed on the list of people I would like to thank. My mother Irmtrud Jurke was and still is a central person supporting me in every imaginable way. The same credit goes to my grandmother Gertrud Angermann. Furthermore, I would like to point out the invaluable help of both my uncles Friedrich Angermann and Helmut Jurke during the relocation to Munich as well as their continued financial support. Still my friends Daniel Altemeier, Cathrin and Sebastian Becker, Michael Brockamp, Martin Menze, Christian Mester and Johannes Röttig are important cornerstones of my life, and I value every minute spent in their company.

Fellow mathematician Fabian Meier proved to be a valuable critic of my work in progress and I would like to thank him for his time. Likewise, I would like to mention Stephan Mescher and his seemingly infinity supply of rather two-dimensional jokes presented at any given opportunity. I'm also indebted to Dr. Andriy Haydys who shared some very helpful insights and ideas to certain problems encountered, as well as the rest of the Bielefeld working group "geometry and topology". Again, my former office mates Florian Kühnel and Daniel Kruppke were always available for either a serious discussion on physics' fundamentals or just plain fun.

As already mentioned in the acknowledgement of my physics' thesis, I learned most of my mathematical knowledge from Prof. Dr. Stefan Bauer and Hsch.-Doz. D.Phil. Kim Frøyshov. In fact, by the time of my third semester a small group of students consisting of Jan Giesselmann, Fabian and Lennart Meier, Otto Kaiser and myself had formed, which followed the lectures of both teachers rather consistently.

In order to describe my thesis advisor, a short story best catches his orginal attitude: After two lecture series on complex analysis by Mr. Frøyshov, certain requirements for introductory courses in the lecturing schedule did not allow for a third semester of complex analysis, which was essentially only relevant for the five aforementioned students. However, after disguising the lecture by the rather innocent name of "complex geometry" right next to the "elementary geometry" reserved for the freshmen, the number of participants was increased from five at the end of "complex analysis 2" to around thirty right before the beginning of the first lecture on "complex geometry". Punctual as ever, Mr. Frøyshov entered the lecture hall after the ten-week summer holidays, grabbed a fresh piece of chalk, took a short look around the classroom and—missing any further introduction—started with the words "Well, last time we introduced the skyscraper sheaf, which was defined by..." while immediately beginning to scribble definitions and theorems to the board. Needless to say, without further effort we were back to the original five people right after the first break about half a hour later.

Gifted with an incredible talent for mathematics, I regularly left his office somewhat more confused afterwards than before entering with a number of questions regarding my current work, only to find out two weeks later that his first ideas and proposals to the problem at hand were quite right and precisely formulated. Therefore, I would like to thank Mr. Frøyshov for his support during the preparation of this thesis and in particular for the suggestion of a mathematical topic that close to theoretical physics.

"Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

"Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture."

Bertrand Russell
(British Philosopher, 1872-1970)

Contents

CHAPTER 1

Introduction

Gauge theory was originally developed from an understanding of Maxwell's theory of electrodynamics in terms of symmetry considerations. In the 1960s and 70s the nature of elementary particles was successfully described using the same concept of local gauge choices but with a much larger symmetry group. Following its success in physics, the mathematician Michael Atiyah began a systematic study of the mathematical properties of the classical Yang-Mills equations around the same time. His student Simon Donaldson later applied gauge theory to the classification of differentiable structures on four-manifolds by finding certain gauge-theoretical invariants bearing his name nowadays. This ultimately led to the discovery of exotic $\mathbb{R}^4$s and similar unexpected results in the mid-80s. More recently, the discovery of the Seiberg-Witten invariants in 1994—mostly due to Witten's study of four-dimensional supersymmetric Yang-Mills theory—renewed the interest in this area of mathematics. Due to the uprising of string theory and the need for suitable higher-dimensional generalizations of the established four-dimensional cases, there is a steadyly growing interaction between theoretical physics and the corresponding mathematics of gauge theory.

Overview of chapters

The central objects in gauge theory are instantons, which are solutions of the Yang-Mills equations that—according to the principle of least action from physics—globally minimize the Yang-Mills functional, i.e. they describe completely stable configurations. From the mathematical point of view, those instantons enjoy a particular simple description using the Hodge star operator, but only in case of four-dimensional base spaces. The basics of differential geometry on principal bundles, which provides the framework of gauge theory, as well as instantons in four-dimensional gauge theory, are introduced in chap. 2.

Given some additional geometrical (and therefore topological) structure, there exists a similar description of instantons in higher dimensions. More precisely, there are two established generalizations: either by complexifying four-dimensional real base spaces and developing a sort of "complex gauge theory" or by restricting to Riemannian manifolds with special holonomy. In the latter approach, one can identify a distinguished subbundle that depends on the Lie algebra of the holonomy subgroup. Requiring the gauge field strength to be a section of this subbundle leads to the generalized instanton equations. Following a short summary of special holonomy, the generalization scheme used to introduce instantons in higher dimensions via special holonomy is presented in chap. 3.

Conversely, one can pass to lower dimensions by imposing certain restrictions on the solutions of the instanton equations, which represent certain symmetries of the given configuration. For example, consider a gauge theory on the product base space $A \times B$. By dropping the gauge field's dependency on the coordinates of B, one effectively describes special solutions of a lower-dimensional gauge theory on A, where the additional components of the original gauge field on $A \times B$ serve as external parameters. This process of dimensional reduction will be carried out in chap. 4 for two cases of Spin(7)-holonomy product manifolds, one being $Z \times \mathbb{R}$, where Z is a G_2-manifold, and the other being $X \times \mathbb{R}^4$ with a K3 surface X.

In the first case, the reduction of the Spin(7) instanton equations on $Z \times \mathbb{R}$ gives a sort of generalized G_2 instanton equation on Z. On $X \times \mathbb{R}^4$ only $\mathbb{R}^4$-independent gauge fields are considered which naturally leads to a gauge theory on X. In fact, the manifold $X \times \mathbb{R}^4$

is identified as the total space of the positive spinor bundle on X, which provides additional structure to the canonical fibration induced by the projection $\mathrm{pr}_1 : X \times \mathbb{R}^4 \longrightarrow\!\!\!\!\rightarrow X$ onto the first factor. In this process of dimensional reduction the remaining components of the gauge field are collected into a so-called Higgs field, which turns out to be a positive spinor. The resulting four-dimensional equations share an intriguing similarity to the well-known Seiberg-Witten equations.

The appendix contains a number of computations and serves mainly as a reference for the main sections. Following a summary of the basic definitions regarding fibre and vector bundles, Clifford algebras and spinors are introduced. In particular, the four-dimensional case and the induced representations of Spin(4) are worked out in detail. In the final section a matrix representation of left $\mathbb{H}$-multiplication is provided.

CHAPTER 2

Classical four-dimensional gauge theory

The purpose of this chapter is to summarize the basics of gauge theory on four-dimensional spaces in order to fix the notation and conventions. Furthermore, the relationship between instantons and Hodge-(anti-)self-duality is investigated. In the next chapter those concepts are generalized to higher-dimensional spaces. Most proofs are only referenced, as the discussed topic is well covered in the literature.

2.1. Differential geometry and gauge theory

The geometrical framework of gauge theories is provided by principal G-bundles, where G is some Lie group. See app. A.2 for some elementary definitions regarding bundles.

Definition 2.1. Let $P \xrightarrow{\pi} M$ be a principal G-bundle with G being a Lie group and $\mathfrak{g}$ its Lie algebra. The **adjoint representation** $\mathrm{ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$ corresponds to the tangent mapping $\mathrm{T}(\mathrm{Ad}_g)_e$ of the **conjugation mapping** $\mathrm{Ad}_g : h \mapsto ghg^{-1}$. This leads to the following definitions:

- The associate fibre bundle of P and G using the conjugation mapping is called the **adjoint fibre bundle** $\mathrm{Ad}\, P \xrightarrow{\pi_{\mathrm{Ad}}} M$, which can be shown to satisfy the conditions of a principal G-bundle.
- The bundle $P \times_{\mathrm{ad}} \mathfrak{g}$ attaches the Lie algebra $\mathfrak{g}$ via the adjoint representation to each point of the base space and it is denoted by $\mathrm{ad}\, P \xrightarrow{\pi_{\mathrm{ad}}} M$. The space of sections $\Gamma(\mathrm{ad}\, P)$ has the structure of a Lie algebra, where the Lie bracket is naturally induced by the pointwise Lie brackets $[.,.]_x$ of each fibre $(\mathrm{ad}\, P)_x \cong \mathfrak{g}$. This bundle is called the **associated Lie algebra bundle** of $P \xrightarrow{\pi} M$.

Definition 2.2. Let $E \xrightarrow{\pi} M$ be a real vector bundle of rank n. Define a new bundle over the same base space with each fibre given by $\mathrm{Fr}(E)_x := \{\text{linear isomorphisms } \mathbb{R}^n \xrightarrow{\cong} E_x\}$, which is isomorphic to $\mathrm{GL}(n;\mathbb{R})$. Equipped with a suitable topology and bundle projection, this gives rise to a principal $\mathrm{GL}(n;\mathbb{R})$-bundle $\mathrm{Fr}(E) \xrightarrow{\pi_{\mathrm{Fr}}} M$, which is called the **frame bundle** of $E \xrightarrow{\pi} M$.

This associates to each vector bundle a corresponding principal bundle. Conversely, given a principal $\mathrm{GL}(n;\mathbb{R})$-bundle, the associated real vector bundle is $E = P \times_\rho \mathbb{R}^n$, using the standard (or fundamental) representation ρ of $\mathrm{GL}(n;\mathbb{R})$. Those considerations naturally extend also to complex vector bundles and principal $\mathrm{GL}(n;\mathbb{C})$-bundles.

Definition 2.3. Let $P \xrightarrow{\pi} M$ be a principal G-bundle and $p \in P$ be a point of the total space. The subspace $V_p := \ker \pi_* \subset \mathrm{T}_p P$ contains the **vertical** vectors, which are tangent to the fibre $P_p \subset P$. A **horizontal distribution** on $P \xrightarrow{\pi} M$ is a smooth choice of subspaces $H_p \subset \mathrm{T}_p P$, such that both $(r_g)_* H_p = H_{pg}$ and $\mathrm{T}_p P = V_p \oplus H_p$ hold for all $p \in P$ and $g \in G$. The vectors in the subspaces H_p are called **horizontal** and any differential form that vanishes on vertical vectors will be called horizontal as well.

Therefore a horizontal distribution on $P \xrightarrow{\pi} M$ is a G-invariant subbundle $H \subset \mathrm{T}P$ complementary to the vertical subbundle $V \subset \mathrm{T}P$. Given a G-invariant Riemannian metric on P, this complement is naturally defined by the orthogonal complement $H_p = V_p^\perp$ with respect to the inner product at each point of the total space.

Definition 2.4. Let $P \xrightarrow{\pi} M$ be a principal G-bundle with the horizontal distribution $H \subset \mathrm{T}P$. For each element $\xi \in \mathfrak{g}$ of the Lie algebra there exists a canonical vertical vector field, called the **fundamental vector field** $\sigma(\xi)$, that is given at each point $p \in P$ by

$$\sigma_p(\xi) := \frac{\mathrm{d}}{\mathrm{d}t}\big(p\exp(t\xi)\big)\Big|_{t=0} . \tag{2.1}$$

The **connection 1-form** associated to H is the $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P;\mathfrak{g})$ defined by

$$\omega(v) := \begin{cases} \xi & : \quad v = \sigma_p(\xi) \text{ is vertical for some } \xi \in \mathfrak{g} \\ 0 & : \quad v \text{ is horizontal} \end{cases} \tag{2.2}$$

for a tangent vector $v \in \mathrm{T}_pP$ at the point $p \in P$ of the bundle's total space.

Proposition 2.5. *Let $P \xrightarrow{\pi} M$ be a principal G-bundle, then the following statements hold:*

(1) *The fundamental vector field satisfies for any $\xi \in \mathfrak{g}$ and $g \in G$*

$$(r_g)_*\sigma(\xi) = \sigma\left(\mathrm{ad}_{g^{-1}}(\xi)\right) . \tag{2.3}$$

(2) *The connection 1-form $\omega \in \Omega^1(P;\mathfrak{g})$ is right G-equivariant, i.e. for all $g \in G$*

$$(r_g)^*\omega = \mathrm{ad}_{g^{-1}}(\omega). \tag{2.4}$$

Proof. The first identity follows from a straightforward computation: For $p \in P$ it follows

$$\begin{aligned}(r_g)_*\sigma_p(\xi) &\overset{(2.1)}{=} \frac{\mathrm{d}}{\mathrm{d}t} r_g\big(p\exp(t\xi)\big)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\big(pgg^{-1}\exp(t\xi)g\big)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(pg\exp\left(t\,\mathrm{ad}_{g^{-1}}(\xi)\right)\Big)\Big|_{t=0} = \sigma_{pg}\left(\mathrm{ad}_{g^{-1}}(\xi)\right) .\end{aligned}$$

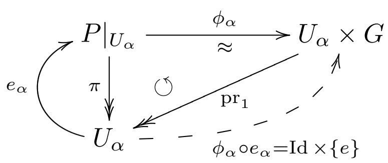
To prove the second statement, let $v \in H_p \subset \mathrm{T}_pP$ be a horizontal vector, i.e. $\omega(v) = 0$. Due to the G-invariance of $H \subset \mathrm{T}P$ it follows $(r_g)_*v \in H_{pg}$, thus $r_g^*\omega(v) = 0$. In the second step let $v = \sigma_p(\xi) \in V_p$ be a vertical vector for some $\xi \in \mathfrak{g}$. It follows

$$r_g^*\omega\big(\sigma(\xi)\big) = \omega\big((r_g)_*\sigma(\xi)\big) \overset{(2.3)}{=} \omega\Big(\sigma\big(\mathrm{ad}_{g^{-1}}(\xi)\big)\Big) = \mathrm{ad}_{g^{-1}}(\xi)$$

from the first statement and the behaviour of the connection 1-form on vertical vectors. □

A horizontal connection $H \subset \mathrm{T}P$ uniquely determines a connection 1-form $\omega \in \Omega^1(P;\mathfrak{g})$. Conversely, given a 1-form $\omega \in \Omega^1(P;\mathfrak{g})$ satisfying (2.4), the associated horizontal distribution $H \subset \mathrm{T}P$ corresponds to the kernel of the connection 1-form, i.e. $H = \ker\omega$. Thus, the notion of a horizontal distribution H and of a connection 1-form ω are equivalent.

Let $\phi := \{\phi_\alpha, U_\alpha\}_{\alpha\in J}$ be a covering collection of local trivializations for the principal G-bundle $P \xrightarrow{\pi} M$, such that $\bigcup_{\alpha\in J} U_\alpha \supseteq M$ holds. Assign to each $\phi_\alpha \in \phi$ a **canonical unit section** $e_\alpha \in \Gamma(P|_{U_\alpha})$ whose value at $x \in U_\alpha \subset M$ is given by



$$\rightsquigarrow \qquad \phi_\alpha \circ e_\alpha(x) = (x, e).$$

Due to the G-equivariance of the connection 1-form $\omega \in \Omega^1(P;\mathfrak{g})$ there is a well-defined local pullback $A_\alpha := e_\alpha^*(\omega|_{U_\alpha}) \in \Omega^1(U_\alpha;\mathfrak{g})$. Let $\theta_g \in \Omega^1(P;\mathfrak{g})$ denote the left-invariant **Maurer-Cartan form** defined by $\theta_g(v) := (\ell_{g^{-1}})_*v$ for $v \in \mathrm{T}_gG$, then two such local pullbacks A_α and A_β are related by virtue of

$$A_\alpha = \mathrm{ad}_{g_{\alpha\beta}} \circ \left(A_\beta - g_{\alpha\beta}^*\theta\right) , \tag{2.5}$$

on domain overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$, see [KN63, prop. 1.4] for a detailed proof. Let $A := \{A_\alpha, U_\alpha\}_{\alpha\in J}$ denote the covering collection of those local pullbacks.

DEFINITION 2.6. A covering collection A of local $\mathfrak{g}$-valued 1-forms satisfying (2.5) is called a **gauge field** on $P \xrightarrow{\pi} M$, particularly in the context of physics.

Thus, there are three equivalent descriptions of the concept of a **connection** on a principal G-bundle $P \xrightarrow{\pi} M$ where G is a Lie group:

- a G-invariant horizontal subbundle $H \subset \mathrm{T}P$,
- a connection 1-form $\omega \in \Omega^1(P;\mathfrak{g})$ as in def. 2.4,
- a gauge field $A = \{A_\alpha, U_\alpha\}_{\alpha\in J}$ satisfying the overlap condition (2.5).

PROPOSITION 2.7. *Any principal G-bundle possesses global connections. The space of connections* $\mathrm{Conn}(P)$ *over the principal G-bundle $P \xrightarrow{\pi} M$ is an infinite-dimensional affine vector space of the form*

$$\mathrm{Conn}(P) = A_0 + \Omega^1(M; \mathrm{ad}\, P).$$

PROOF. The existence of connections is shown in [KN63, sec. II.2]. In order to show the second statement, let $\omega, \omega' \in \Omega^1(P;\mathfrak{g})$ be two connection 1-forms on $P \xrightarrow{\pi} M$. Since both ω and ω' agree on vertical vectors, the difference $\rho := \omega - \omega'$ is a horizontal 1-form satisfying the G-equivariance condition, i.e. $(r_g)^*\rho = \mathrm{ad}_{g^{-1}}(\rho)$. Given a covering collection of local trivializations $\phi = \{\phi_\alpha, U_\alpha\}_{\alpha\in J}$ and corresponding local unit sections $\{e_\alpha, U_\alpha\}_{\alpha\in J}$, let $\rho_\alpha := e_\alpha^*(\tau|_{U_\alpha}) \in \Omega^1(U_\alpha;\mathfrak{g})$ denote the corresponding pullbacks. From (2.5) it follows

$$\begin{aligned}\rho_\alpha = A_\alpha - A'_\alpha &= \mathrm{ad}_{g_{\alpha\beta}} \circ \left(A_\beta - g^*_{\alpha\beta}\theta\right) - \mathrm{ad}_{g_{\alpha\beta}} \circ \left(A'_\beta - g^*_{\alpha\beta}\theta\right)\\ &= \mathrm{ad}_{g_{\alpha\beta}} \circ \left(A_\beta - A'_\beta\right) = \mathrm{ad}_{g_{\alpha\beta}} \circ \rho_\beta,\end{aligned}$$

on non-empty domain overlaps $U_\alpha \cap U_\beta \neq \emptyset$, such that the covering collection $\{\rho_\alpha, U_\alpha\}_{\alpha\in J}$ of local pullbacks $\rho_\alpha \in \Omega^1(U_\alpha;\mathfrak{g})$ gives rise to a global 1-form $\rho \in \Omega^1(M; \mathrm{ad}\, P)$. □

Let $P \xrightarrow{\pi} M$ be a principal G-bundle with the horizontal subbundle $H \subset \mathrm{T}P$. The **horizontal projection** $h : \mathrm{T}P \twoheadrightarrow H \subset \mathrm{T}P$ is pointwise defined by

$$h_p(v) := \begin{cases} v & : \quad v \in H_p \text{ (horizontal vector)} \\ 0 & : \quad v \in V_p \text{ (vertical vector)} \end{cases}.$$

The dual mapping $\tilde{h} : \mathrm{T}^*P \longrightarrow \mathrm{T}^*P$ is given by $\tilde{h}(\alpha) = \alpha \circ h$ for any $\alpha \in \Omega^1(P)$ and naturally extends to a mapping on k-forms. Furthermore, a $\mathfrak{g}$-valued k-form $\alpha \in \Omega^k(P;\mathfrak{g})$ is said to be **basic** if it is both horizontal (i.e. $\tilde{h}(\alpha) = \alpha$) and G-equivariant (i.e. $(r_g)^*\alpha = \mathrm{ad}_{g^{-1}}(\alpha)$ holds).

PROPOSITION 2.8. *The space of basic forms $\Omega^k_{\mathrm{b}}(P;\mathfrak{g})$ is isomorphic to $\Omega^k(M; \mathrm{ad}\, P)$.*

PROOF. See [Fig06, sec. 2.2.1] for a construction of the required isomorphism. □

DEFINITION 2.9. Let $P \xrightarrow{\pi} M$ be a principal G-bundle. The **exterior covariant derivative** $\mathrm{d}^H : \Omega^k_{\mathrm{b}}(P;\mathfrak{g}) \longrightarrow \Omega^{k+1}_{\mathrm{b}}(P;\mathfrak{g})$ is defined by $\mathrm{d}^H\alpha := \tilde{h}(\mathrm{d}\alpha)$. The same notion also refers to the corresponding mapping $\mathrm{d}_A : \Omega^k(M; \mathrm{ad}\, P) \longrightarrow \Omega^{k+1}(M; \mathrm{ad}\, P)$ by virtue of the previous isomorphism.

The adjoint representation $\mathrm{ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$ induces a representation of the Lie algebra

$$\begin{aligned}\mathfrak{ad} := \mathrm{T}(\mathrm{ad})_e : \mathfrak{g} &\longrightarrow \mathrm{End}(\mathfrak{g})\\ \xi &\mapsto [\xi, .]\end{aligned}$$

provided by the Lie bracket, which can be used to describe d^H or d_A explicitly.

PROPOSITION 2.10. *Let $P \xrightarrow{\pi} M$ be a gauge field, then the exterior covariant derivative can be described as follows:*

(1) *Let $\omega \in \Omega^1(P;\mathfrak{g})$ be the connection 1-form and $\phi \in \Omega^k_{\mathrm{b}}(P;\mathfrak{g})$ a basic k-form. Then the exterior covariant derivative is given by*

$$\mathrm{d}^H\phi = \mathrm{d}\phi + \mathfrak{ad}_\omega \wedge \phi,$$

where the "wedge" $\wedge$ denotes both the wedge product of differential forms and the composition of components via $\mathfrak{ad}$. *For a 0-form (i.e. function* $P \longrightarrow \mathfrak{g}$*) the general formula simplifies to* $(\mathrm{d}^H\phi)(X) = \mathrm{d}\phi(X) + \mathfrak{ad}_{\omega(X)} \circ \phi$ *for some vector field* $X \in \mathfrak{X}(P)$.

(2) *Let* $A = \{A_\alpha, U_\alpha\}_{\alpha \in J}$ *be a gauge field. Any k-form* $\psi \in \Omega^k(M; \mathrm{ad}\, P)$ *is described by a covering collection* $\psi = \{\psi_\alpha, U_\alpha\}_{\alpha \in J}$ *of local pullbacks* $\psi_\alpha := e_\alpha^*(\psi) \in \Omega^k(U_\alpha; \mathfrak{g})$. *The exterior covariant derivative is given by*

$$\mathrm{d}_A\psi_\alpha = \mathrm{d}\psi_\alpha + \mathfrak{ad}_{A_\alpha} \wedge \psi_\alpha,$$

which for a 0-form simplifies to $(\mathrm{d}_A\psi_\alpha)(Y) = \mathrm{d}\psi_\alpha(Y) + \mathfrak{ad}_{A_\alpha(Y)} \circ \psi_\alpha$ *for some restriction* $Y := X|_{U_\alpha}$ *of the vector field* $X \in \mathfrak{X}(M)$.

PROOF. For example, see the exposition in [Fig06, sec. 2.2.2]. □

DEFINITION 2.11. Let $P \xrightarrow{\pi} M$ be a principal G-bundle with the connection $H \subset \mathrm{T}P$ and the corresponding connection 1-form $\omega \in \Omega^1(P; \mathfrak{g})$. Then $\Omega := \tilde{h}(\mathrm{d}^H\omega) \in \Omega^2(P; \mathfrak{g})$ is called the **curvature 2-form** of the connection.

Due to $\tilde{h}\omega = 0$ for a connection 1-form, there is the identity $\Omega(U, V) = -\omega\big([hU, hV]_{\mathfrak{X}(P)}\big)$, i.e. the 2-form measures the **integrability** of the horizontal subbundle $H \subset \mathrm{T}P$, i.e. whether the Lie bracket of any two sections of the subbundle $H \subset \mathrm{T}P$ lies again in H. Let now $[.\wedge.]$ denote the symmetric bilinear product which links the ordinary wedge product and the Lie bracket $[.,.]_{\mathfrak{g}}$ of $\mathfrak{g}$.

PROPOSITION 2.12. *Let* $P \xrightarrow{\pi} M$ *be a principal G-bundle with* $\omega \in \Omega^1(P; \mathfrak{g})$ *being the connection 1-form. Then the following two identities hold:*

(1) *Structure equation:* $\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega]$

(2) *Bianchi identity:* $\mathrm{d}^H\Omega = \tilde{h}(\mathrm{d}\Omega) = 0$

PROOF. See [KN63, thm. II.5.2] and [KN63, thm. II.5.4]. □

DEFINITION 2.13. Let $\Omega \in \Omega^2(P; \mathfrak{g})$ be a curvature 2-form and $\phi = \{\phi_\alpha, U_\alpha\}_{\alpha \in J}$ a covering collection of local trivializations with corresponding local unit sections $\{e_\alpha, U_\alpha\}_{\alpha \in J}$. Define the local curvature pullbacks $F_\alpha := e_\alpha^*(\Omega|_{U_\alpha}) \in \Omega^2(U_\alpha; \mathfrak{g})$. By virtue of the structure equation it follows $F_\alpha = \mathrm{ad}_{g_{\alpha\beta}} \circ F_\beta$ on non-empty domain overlaps $U_{\alpha\beta}$, such that the covering collection $F = \{F_\alpha, U_\alpha\}_{\alpha \in J}$ gives rise to the **gauge field strength** $F \in \Omega^2(M; \mathrm{ad}\, P)$. The notation F_A for the gauge field strength indicates the dependency on the gauge field (i.e. connection) A.

The gauge field strength is an equivalent description of curvature. Whereas the curvature 2-form $\omega \in \Omega^2(P; \mathfrak{g})$ and horizontal distribution $H \subset \mathrm{T}P$ are defined on the total space of the principal G-bundle, the gauge field strength $F \in \Omega^2(M; \mathrm{ad}\, P)$ is a differential form on the base space that inherits the bundle's topology via the adjoint bundle.

PROPOSITION 2.14. *Let* $P \xrightarrow{\pi} M$ *be a principal G-bundle and* $A = \{A_\alpha, U_\alpha\}_{\alpha \in J}$ *a gauge field on P. Then the two statements of prop. 2.12 are equivalent to:*

(1) *Structure equation:* $F_\alpha = \mathrm{d}A_\alpha + \frac{1}{2}[A_\alpha \wedge A_\alpha]$

(2) *Bianchi identity:* $\mathrm{d}_A F_A = 0.$

DEFINITION 2.15. Let $P \xrightarrow{\pi} M$ be principal G-bundle. A **gauge transformation** is G-equivariant diffeomorphism $\Psi : P \xrightarrow{\approx} P$ of the bundle's total space that preserves the fibres, i.e. it makes the following diagram commutative:

$$\begin{array}{ccc} P & \xrightarrow[\approx]{\Psi} & P \\ & {\scriptstyle\pi}\searrow \; \circlearrowleft \; \swarrow{\scriptstyle\pi} & \\ & M & \end{array} \qquad \rightsquigarrow \qquad (\Psi P)_x = P_x \text{ as sets.}$$

A gauge transformation can be understood as the natural notion of an automorphism for a principal G-bundle. However, as a connection on $P \xrightarrow{\pi} M$ will usually be changed by a gauge transformation, one has to check any connection-dependent quantities for gauge invariance.

2.2. The Yang-Mills functional

In physics the dynamics of any object is governed by the principle of least action. For the case at hand—pure gauge theory—this leads to the Yang-Mills equations, whose solutions minimize the Yang-Mills functional.

DEFINITION 2.16. Let G be a Lie group and $\mathfrak{g}$ the Lie algebra with its natural adjoint action. The **Killing form** $\kappa(\xi,\zeta) := \operatorname{Tr} \mathfrak{ad}_\xi\, \mathfrak{ad}_\zeta = \operatorname{Tr}\big[\xi,[\zeta,.]\big]$ is an inner product on the Lie algebra $\mathfrak{g}$, which is invariant under the adjoint action of G.

Let $\phi,\psi \in \Omega^k(M;\operatorname{ad}P)$ be two k-forms with values in the adjoint Lie algebra bundle, which are represented by covering collections $\phi = \{\phi_\alpha, U_\alpha\}_{\alpha\in J}$ and $\psi = \{\psi_\alpha, U_\alpha\}_{\alpha\in J}$ of local pullbacks $\phi_\alpha, \psi_\alpha \in \Omega^k(M;\mathfrak{g})$ with respect to some trivialization $\phi = \{\phi_\alpha, U_\alpha\}_{\alpha\in J}$ and its local unit sections $\{e_\alpha, U_\alpha\}_{\alpha\in J}$. Then by virtue of

$$\kappa(\phi_\alpha,\psi_\alpha) = \kappa\big(\operatorname{ad}_{g_{\alpha\beta}}(\phi_\beta), \operatorname{ad}_{g_{\alpha\beta}}(\psi_\beta)\big) = \kappa(\phi_\beta,\psi_\beta)$$

on non-empty domain overlaps, $U_\alpha \cap U_\beta \neq \emptyset$, the Killing form gives rise to a pointwise adjoint-invariant inner product $\langle .,.\rangle_{\mathrm{ad}}$ on the space $\Omega^k(M;\operatorname{ad}P)$.

DEFINITION 2.17. Let $P \xrightarrow{\pi} M$ be a principal G-bundle with the gauge field A and the gauge field strength $F_A \in \Omega^2(M;\operatorname{ad}P)$. The **Yang-Mills functional** is defined by

$$\begin{aligned} S_{\mathrm{YM}} &: \operatorname{Conn}(P) \longrightarrow \mathbb{R} \\ A &\mapsto S_{\mathrm{YM}}(A) := \int_M \|F_A\|^2 \,\mathrm{dvol}_M, \end{aligned}$$

where the norm $\|F_A\|^2 := \langle F_A, F_A\rangle_{\mathrm{ad}} \in C^\infty(M)$ is induced by the inner product on $\Omega^k(M;\operatorname{ad}P)$.

PROPOSITION 2.18. *The Yang-Mills functional does not depend on the local unit section* $e_\alpha \in \Gamma(P|_{U_\alpha})$ *used for the local pullbacks of the curvature 2-form.*

PROOF. Since both the horizontal and vertical subbundles $H, V \subset \mathrm{T}P$ are G-invariant, it follows $h \circ (r_g)_* = (r_g)_* \circ h$ for any $g \in G$, where $h : \mathrm{T}P \twoheadrightarrow H$ is the horizontal projection. Using the structure equation and (2.4), this leads to

$$(r_g)^*\Omega = \operatorname{ad}_{g^{-1}} \circ\, \Omega. \tag{2.6}$$

After a refinement of the coverings, one can assume $\{e_\alpha, U_\alpha\}_{\alpha\in J}$ and $\{\tilde e_\alpha, U_\alpha\}_{\alpha\in J}$ to be two sets of local unit sections with same domains. Due to the transitive action of G on each fibre of $P \xrightarrow{\pi} M$, there exists a collection $\{h_\alpha, U_\alpha\}_{\alpha\in J}$ of functions $h_\alpha : U_\alpha \longrightarrow G$, such that the relation $\tilde e_\alpha(x) = e_\alpha(x) h_\alpha(x)$ holds for all $x \in U_\alpha$ for each $\alpha \in J$.

Let $F_\alpha := e_\alpha^*(\Omega|_{U_\alpha})$ and $\tilde F_\alpha := \tilde e_\alpha^*(\Omega|_{U_\alpha})$ denote the local pullbacks of the curvature 2-form $\Omega \in \Omega^2(P;\mathfrak{g})$. To ease up readability, the index "α" will be dropped, and it follows

$$\begin{aligned} \tilde F_x &= (\tilde e^*\Omega)_x = \big((r_h \circ e)^*\Omega\big)_x = \big(e^*(r_h)^*\Omega\big)_x \\ &\overset{(2.6)}{=} \big(e^*(\operatorname{ad}_{h^{-1}} \circ\, \Omega)\big)_x = \operatorname{ad}_{h^{-1}(x)} \circ (e^*\Omega)_x = \operatorname{ad}_{h^{-1}(x)} \circ F_x. \end{aligned}$$

The invariance of the inner product by the adjoint G-action implies $\|F_A\|^2 = \|\tilde F_A\|^2$. □

LEMMA 2.19. *Let* $P \xrightarrow{\pi} M$ *be a principal* G*-bundle and* $\Psi : P \xrightarrow{\approx} P$ *be a gauge transformation. The connection and curvature are changed as follows:*

$$\begin{aligned} \omega &\mapsto \omega^\Psi = (\Psi^{-1})^*\omega && \textit{(connection 1-form)} \\ \Omega &\mapsto \Omega^\Psi = (\Psi^{-1})^*\Omega && \textit{(curvature 2-form)} \end{aligned}$$

PROOF. Let $H, V \subset \mathrm{T}P$ be the horizontal and vertical subbundle, which transform as tangent spaces to $H^\Psi = \Psi_* H$ and $V^\Psi = \Psi_* V$. Using G-equivariance of the gauge transformation, it follows

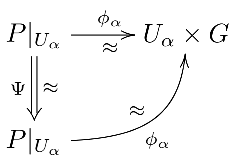
$$\sigma_p(\xi) = \frac{\mathrm{d}}{\mathrm{d}t} p\mathrm{e}^{t\xi}\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \Psi \circ \Psi^{-1}(p)\exp(t\xi)\Big|_{t=0} = \Psi_*\sigma_{\Psi^{-1}(p)}(\xi)$$

for the fundamental vertical vector field. Let $\omega \in \Omega^1(P;\mathfrak{g})$ be the connection 1-form to the horizontal distribution, then it follows for some tangent vector $v \in \mathrm{T}_pP$

$$\begin{aligned}(\omega^\Psi)_p(v) &= \begin{cases} \xi & : \quad v \in (V^\Psi)_p = \Psi_* V_{\Psi^{-1}(p)} \text{ such that } \sigma_p(\xi) = \Psi_*\sigma_{\Psi^{-1}(p)}(\xi) = v \\ 0 & : \quad v \in (H^\Psi)_p = \Psi_* H_{\Psi^{-1}(p)} \end{cases} \\ &= \begin{cases} \xi & : \quad (\Psi_*)^{-1}v \in V_{\Psi^{-1}(p)} \text{ such that } \sigma_{\Psi^{-1}(p)}(\xi) = (\Psi_*)^{-1}v \\ 0 & : \quad (\Psi_*)^{-1}v \in H_{\Psi^{-1}(p)} \end{cases} \\ &= \omega_{\Psi^{-1}(p)}\big((\Psi_*)^{-1}v\big) \\ &= (\Psi^{-1})^*\omega_p(v),\end{aligned}$$

which proves the statement for connection 1-forms. The second statement for connection 2-forms immediately follows from the structure equation in prop. 2.12. □

Remark 2.20. Let $\phi_\alpha : P|_{U_\alpha} \xrightarrow{\approx} U_\alpha \times G$ be a local trivialization of the principal G-bundle $P \xrightarrow{\pi} M$, which can be represented as $\phi_\alpha(p) = \big(\pi(p), g_\alpha(p)\big)$ for some continuous G-equivariant function $g_\alpha : P|_{U_\alpha} \longrightarrow G$. Following this scheme a gauge transformation $\Psi : P \xrightarrow{\approx} P$ is locally described by



$$\rightsquigarrow \qquad \begin{aligned}\phi_\alpha\big(\Psi(p)\big) &= \Big(\pi\big(\Psi(p)\big), g_\alpha\big(\Psi(p)\big)\Big) \\ &= \Big(\pi(p), g_\alpha\big(\Psi(p)\big)\Big).\end{aligned}$$

Define a mapping $\bar{\psi}_\alpha : P|_{U_\alpha} \longrightarrow G$ by $\bar{\psi}_\alpha(p) := g_\alpha(p)\bar{\Psi}(p)g_\alpha(p)^{-1}$. Due to the G-equivariance of g_α and Ψ this mapping $\bar{\psi}_\alpha$ does not depend on the fibre, i.e. there exists a continuous mapping $\psi_\alpha : U_\alpha \longrightarrow G$ such that $\bar{\psi}_\alpha(p) = \psi_\alpha\big(\pi(p)\big)$. It follows

$$\phi_\alpha\big(\Psi(p)\big) = \Big(\pi(p), \psi_\alpha\big(\pi(p)\big)g_\alpha(p)\Big),$$

hence a gauge transformation corresponds to a G-action that only depends on the base space.

Proposition 2.21. *The Yang-Mills functional is gauge-invariant.*

Proof. By [Fig06, prop. 1.3] the restriction of the connection 1-form $\omega \in \Omega^1(P;\mathfrak{g})$ to $P|_{U_\alpha}$ agrees with $\mathrm{ad}_{g_\alpha^{-1}} \circ \pi^* A_\alpha + g_\alpha^*\theta$, which gives an explicit description of $\omega|_{P|_{U_\alpha}} = (e_\alpha^*)^{-1}A_\alpha$. The connection 1-form's gauge transformation behavior then translates to

$$A_\alpha \mapsto A_\alpha^\Psi = \mathrm{ad}_{\psi_\alpha} \circ (A_\alpha - \psi_\alpha^*\theta)\,,$$

as shown in [Fig06, sec. 1.6]. This implies $F_\alpha \mapsto F_\alpha^\Psi = \mathrm{ad}_{\psi_\alpha} \circ F_\alpha$ for the gauge field strength. Due to the inner product's G-invariance in the adjoint representation, it follows $\|F_A^\Psi\| = \|F_A\|$, which provides the gauge invariance of the Yang-Mills functional. □

Definition 2.22. Let $P \xrightarrow{\pi} M$ be a principal G-bundle. A critical point $A \in \mathrm{Conn}(P)$ of the Yang-Mills functional $S_{\mathrm{YM}}(A)$ is called a **Yang-Mills connection** and $\mathrm{Conn}^{\mathrm{YM}}(P)$ denotes the space of Yang-Mills connections.

Due to the gauge-invariance, the Yang-Mills functional descents to a mapping on the quotient space $\mathrm{Conn}(P)/\mathrm{Aut}(P)$. Furthermore, any gauge transformation applied to a Yang-Mills connection gives another Yang-Mills connection.[a]

Proposition 2.23. *Let $P \xrightarrow{\pi} M$ be a principal G-bundle. A connection A on P is a Yang-Mills connection if the* ***Yang-Mills equation*** $\mathrm{d}_A^* F_A = 0$ *is satisfied.*

[a]In the context of physics one refers to $\mathrm{Conn}^{\mathrm{YM}}(P)/\mathrm{Aut}(P)$ as the space of **classical solutions**.

PROOF. According to prop. 2.7 the space of connections is an affine vector space. Let $\alpha \in \Omega^1(M; \mathrm{ad}\, P)$ be a 1-form and $A \in \mathrm{Conn}(P)$ a connection. By the structure equation it follows

$$\begin{aligned} F_{A+t\alpha} &= F_A + t\mathrm{d}_A\alpha + \frac{1}{2}t^2[\alpha,\alpha] \\ \Rightarrow \|F_{A+t\alpha}\|^2 &= \|F_A\|^2 + t^2\|\mathrm{d}_A\alpha\|^2 + \frac{1}{4}t^4\|[\alpha,\alpha]\| \\ &\quad + 2t\langle F_A, \mathrm{d}_A\alpha\rangle_{\mathrm{ad}} + t^2\langle F_A, [\alpha,\alpha]\rangle_{\mathrm{ad}} + t^3\langle \mathrm{d}_A\alpha, [\alpha,\alpha]\rangle_{\mathrm{ad}}. \end{aligned}$$

The condition for a critical point of the Yang-Mills function is then restated as follows:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t} S_{\mathrm{YM}}(A+t\alpha)\Big|_{t=0} = \int_M \frac{\mathrm{d}}{\mathrm{d}t}\|F_{A+t\alpha}\|^2\Big|_{t=0} \mathrm{dvol} \\ &= \int_M 2\langle F_A, \mathrm{d}_A\alpha\rangle\, \mathrm{dvol} = 2\int_M \langle \mathrm{d}_A^* F_A, \alpha\rangle\, \mathrm{dvol} \end{aligned}$$

only holds for all $\alpha \in \Omega^1(M; \mathrm{ad}\, P)$ if $\mathrm{d}_A^* F_A = 0$, which proves the statement. □

By definition a **harmonic *p*-form** satisfies $\Delta\alpha = 0$, where $\Delta := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ is the **Hodge Laplacian**. Using the previous result and the Bianchi identity it follows $\Delta F_A = 0$ for any $A \in \mathrm{Conn}^{\mathrm{YM}}(P)$.

2.3. Instantons in four-dimensional gauge theory

The previously introduced Yang-Mills connections are the critical points of the Yang-Mills functional, i.e. the corresponding physical configurations satisfy the equations of motion. However, from the physical point of view, only the global minima describe fully stable configurations, which are called instantons.

DEFINITION 2.24. Let (M, g) be an oriented Riemannian manifold of dimension n, where the metric g provides an inner product $\langle .,. \rangle_g := g_p : \odot^2 \mathrm{T}_p M \longrightarrow \mathbb{R}$ on each tangent space. This canonically extends to an inner product on tensors and k-forms. The **Hodge star operator** $\star : \Lambda^k(M) \xrightarrow{\cong} \Lambda^{n-k}(M)$ is defined by the following property: For any k-form $\beta \in \Omega^k(M)$ the Hodge dual $\star\beta \in \Omega^{n-k}(M)$ is the uniquely given $(n-k)$-form such that $\alpha \wedge \star\beta = \langle \alpha, \beta\rangle_g\, \mathrm{dvol}_M$ holds for all k-forms $\alpha \in \Omega^k(M)$.

REMARK 2.25. The Hodge star operator satisfies $\star(\star\beta) = (-1)^{k(n-k)}\,\mathrm{Id}$ for any k-form $\beta \in \Omega^k(M)$. The inverse is thus given by $\star^{-1} = (-1)^{k(n-k)}\star$.

DEFINITION 2.26. Let (M, g) be an oriented Riemannian manifold of dimension $4m$. Due to $(\star_{2m})^2 = \mathrm{Id}$ there is a ± 1-eigenspace splitting $\Lambda^{2m}_+(M) \oplus \Lambda^{2m}_-(M)$. The corresponding sections are called **self-dual** and **anti-self-dual**.

The following identities regarding (anti-)self-dual differential forms will become important in subsequent sections:

$$\begin{aligned} \alpha_+ \wedge \alpha_+ &= \alpha_+ \wedge \star\alpha_+ = \langle \alpha_+, \alpha_+\rangle\, \mathrm{dvol} = \|\alpha_+\|^2\, \mathrm{dvol} \\ \alpha_- \wedge \alpha_- &= -\alpha_- \wedge \star\alpha_- = -\langle \alpha_-, \alpha_-\rangle\, \mathrm{dvol} = -\|\alpha_-\|^2\, \mathrm{dvol} \end{aligned} \tag{2.7}$$

Furthermore, by the symmetry of the inner product and the symmetry of the wedge product on m-forms with m even, the identity

$$\begin{aligned} \alpha_+ \wedge \alpha_- &= -\alpha_+ \wedge \star\alpha_- = -\langle \alpha_+, \alpha_-\rangle\, \mathrm{dvol} = -\langle \alpha_-, \alpha_+\rangle\, \mathrm{dvol} \\ &= -\alpha_- \wedge \star\alpha_+ = -\alpha_- \wedge \alpha_+ = -\alpha_+ \wedge \alpha_- \end{aligned}$$

implies both $\langle \alpha_+, \alpha_-\rangle = 0$ and $\alpha_+ \wedge \alpha_- = 0$. The importance of the four-dimensional case comes from the fact that the gauge field strength is a 2-form $F_A \in \Omega^2(M; \mathrm{ad}\, P)$ and is therefore subject to the splitting into self-dual and anti-self-dual parts.

Theorem 2.27. *Let $P \xrightarrow{\pi} M$ be a principal G-bundle over a four-dimensional oriented Riemannian manifold (M,g). The Yang-Mills functional can be reformulated to*

$$S_{\mathrm{YM}}(A) = \Xi + 2\int_M \|F_A^-\|^2 \,\mathrm{dvol},$$

for any $A \in \mathrm{Conn}(P)$, where Ξ is a topological invariant of the bundle.

Proof. Let $A \in \mathrm{Conn}(P)$ a connection and $F_A = F_A^+ + F_A^- \in \Omega^2(M; \mathrm{ad}\, P)$ be the decomposed gauge field strength. Due to $\langle F_A^+, F_A^- \rangle_{\mathrm{ad}} = 0$ it follows

$$S_{\mathrm{YM}}(A) = \int_M \|F_A^+ + F_A^-\|^2 \,\mathrm{dvol} = \int_M \|F_A^+\|^2 \,\mathrm{dvol} + \int_M \|F_A^-\|^2 \,\mathrm{dvol}$$

for the Yang-Mills functional when expressed in self-dual and anti-self-dual parts of the gauge field strength. By definition of the inner product on $\Omega^4(M; \mathrm{ad}\, P)$ the identity

$$\begin{aligned} \mathrm{Tr}\, F_A \wedge F_A &= \mathrm{Tr}\, F_A^+ \wedge F_A^+ + \mathrm{Tr}\, F_A^+ \wedge F_A^- + \mathrm{Tr}\, F_A^- \wedge F_A^+ + \mathrm{Tr}\, F_A^- \wedge F_A^- \\ &= \left(\|F_A^+\|^2 + 2\langle F_A^+, F_A^- \rangle_{\mathrm{ad}} - \|F_A^-\|^2 \right) \mathrm{dvol} = \left(\|F_A^+\|^2 - \|F_A^-\|^2 \right) \mathrm{dvol} \end{aligned}$$

holds, such that the Yang-Mills functional can be rewritten to

$$S_{\mathrm{YM}}(A) = \int_M \mathrm{Tr}\, F_A \wedge F_A + 2\int_M \|F_A^-\|^2 \,\mathrm{dvol}.$$

By means of the Chern-Weil homomorphism the first term corresponds to a characteristic class, more explicitly to the first Pontrjagin number

$$\Xi = \int_M \mathrm{Tr}\, F_A \wedge F_A = 4\pi^2 p_1(\mathrm{ad}\, P)[M],$$

as shown in [KN69, sec. XII.4] or [MS74, p. 308]. Thus, Ξ only depends on the topology of $\mathrm{ad}\, P \xrightarrow{\tilde{\pi}} M$ and therefore ultimately on the topology of $P \xrightarrow{\pi} M$. □

Further information on the characteristic class appearing in Ξ and the Chern-Weil homomorphism is found in [DK90, sec. 2.1.4]. Replacing $\Xi = \int_M \left(\|F_A^+\|^2 - \|F_A^-\|^2 \right) \mathrm{dvol}$ by its negative and following the steps of the previous proof gives

$$S_{\mathrm{YM}}(A) = -\Xi + 2\int_M \|F_A^+\|^2 \,\mathrm{dvol}.$$

Obviously, depending on the sign of Ξ, the Yang-Mills action is minimized for either $F_A^+ = 0$ or $F_A^- = 0$, i.e. if the gauge field strength is either self-dual or anti-self-dual.

Definition 2.28. Let $P \xrightarrow{\pi} M$ be a principal G-bundle over a four-dimensional oriented Riemannian manifold M. A gauge field $A \in \mathrm{Conn}(P)$ is called an **(anti-)self-dual instanton** if the gauge field strength $F_A = F_A^{\pm} \in \Omega^2(M; \mathrm{ad}\, P)$ is an (anti-)self-dual 2-form.

Lemma 2.29. *Any such (anti-)self-dual instanton satisfies the Yang-Mills equations.*

Proof. Let $A \in \mathrm{Conn}(P)$ be a gauge field, such that $F_A \in \Omega^2(M; \mathrm{ad}\, P)$ is either self-dual or anti-self-dual. The formal adjoint of the covariant exterior derivative is explicitly given by $\mathrm{d}_A^* = (-1)^{kn+n+1} \star \mathrm{d}_A \star$ for a k-form on a n-dimensional manifold. By virtue of the Bianchi identity it follows $\mathrm{d}_A^* F_A^{\pm} = -\star \mathrm{d}_A \star F_A^{\pm} = \mp \star \mathrm{d}_A F_A^{\pm} = 0$, which implies $A \in \mathrm{Conn}^{\mathrm{YM}}(P)$. □

It is noteworthy to appreciate the particular simple nature of four-dimensional gauge instantons: The gauge field strength depends on the gauge field by means of a first-order differential operator. The instanton equations $F_A^{\pm} = 0$ are therefore described by a first-order partial differential equation. The general Yang-Mills equation $\mathrm{d}_A^* F_A = 0$ on the other hand is a second-order partial differential equation.

2.4. Dimensional reduction in four dimensions

Within the current framework of n-dimensional gauge theory, (anti-)self-dual instantons in four dimensions are the only class of solutions that is truly singled out from the rest. However, one may consider solutions satisfying certain additional conditions. For example, take an instanton on a four-manifold $Y^3 \times \mathbb{R}$ that is invariant under translations in the $\mathbb{R}$-direction. This describes distinguished gauge field in lower dimensions, which is obtained by the process of **dimensional reduction**.

DEFINITION 2.30. Let $M := X^k \times \mathbb{R}^{n-k}$ be a n-dimensional product space, where X is an k-dimensional manifold and $\mathbb{R}^{n-k}$ the usual Euclidean $(n-k)$-plane. Furthermore, let $\mathrm{pr}_1 : M \twoheadrightarrow X$ denote the projection on the first factor X. Define a translation mapping

$$\begin{aligned} T_v : X \times \mathbb{R}^{n-k} &\longrightarrow X \times \mathbb{R}^{n-k} \\ (x,y) &\mapsto (x, y+v). \end{aligned}$$

Let $P \xrightarrow{\pi} X$ be a principal G-bundle and $\mathrm{pr}_1^* P \xrightarrow{\tilde{\pi}} M$ be the pullback bundle. A gauge field A on $\mathrm{pr}_1^* P$ is said to be **translation-invariant** if $T_v^* A = A$ holds for all $v \in \mathbb{R}^{n-k}$.

EXAMPLE 2.31. Let $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ be a four-dimensional product space and $P \xrightarrow{\pi} \mathbb{R}^4$ be a principal G-bundle. Since $H^k(\mathbb{R}^4) \cong \{0\}$ for all $k > 0$ all characteristic classes vanish, such that the bundle is trivial and can be taken as $P = \underline{G} := \mathbb{R}^4 \times G \xrightarrow{\mathrm{pr}_1} \mathbb{R}^4$ where G is the bundle group. Then the adjoint vector bundle $\mathrm{ad}\, P = \underline{\mathfrak{g}} := \mathbb{R}^4 \times \mathfrak{g} \xrightarrow{\mathrm{pr}_1} \mathbb{R}^4$ is also trivial, and the space of connections has the form $\mathrm{Conn}(P) = \mathrm{d} + \Omega^1(\mathbb{R}^4; \mathfrak{g})$ with d serving as the reference connection. Thus, any connection (or rather gauge field) is given by a 1-form $A \in \Omega^1(M; \mathfrak{g})$ and takes the form

$$A = \sum_{i=1}^{4} A_i(x_1, \ldots, x_4)\, \mathrm{d}x^i$$

with respect to local coordinates $(x_1, \ldots, x_4)$ for $\mathbb{R}^4$, where each $A_i \in \Gamma(\underline{\mathfrak{g}}) = C^\infty(\mathbb{R}^4, \mathfrak{g})$ is a component function.

Assume $\tilde{A}$ is a translation-invariant gauge field on $P \xrightarrow{\pi} \mathbb{R}^4$, which means it is independent of the fourth coordinate x_4. In local coordinates the gauge field can be seperated into

$$\tilde{A} = \sum_{i=1}^{3} A_i(x_1, \ldots, x_3)\, \mathrm{d}x^i + A_4(x_1, \ldots, x_3)\, \mathrm{d}x^4 =: A' + \Phi,$$

where A' gives a connection on $\mathbb{R}^3 = \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$ and Φ is called the **Higgs field**. Using this splitting and the structure equation, the gauge field strength is

$$F_{\tilde{A}} = F_{A'+\Phi} = F_{A'} + \mathrm{d}_{A'}\Phi + \frac{1}{2}[\Phi \wedge \Phi] = F_{A'} + \mathrm{d}\Phi$$

and the self-dual instanton equation becomes $F_{A'}^+ = -\pi_+(\mathrm{d}\Phi)$. The basis of $\Lambda_+^2 \mathbb{R}^4$ and the projection $\pi_+ : \Lambda^2\mathbb{R}^4 \twoheadrightarrow \Lambda_+^2 \mathbb{R}^4$ is explicitly given by

$$\begin{aligned} \rho_1^+ &:= \mathrm{d}x^{12} + \mathrm{d}x^{13} & & \pi_+(\mathrm{d}x^{14}) = \pi_+(\mathrm{d}x^{23}) = \rho_3^+ \\ \rho_2^+ &:= \mathrm{d}x^{13} + \mathrm{d}x^{42} & \rightsquigarrow \quad & \pi_+(\mathrm{d}x^{24}) = \pi_+(\mathrm{d}x^{31}) = -\rho_2^+ \\ \rho_3^+ &:= \mathrm{d}x^{14} + \mathrm{d}x^{23} & & \pi_+(\mathrm{d}x^{34}) = \pi_+(\mathrm{d}x^{12}) = \rho_1^+, \end{aligned}$$

and the differential of the Higgs field can be evaluated to

$$\begin{aligned} \pi_+(\mathrm{d}\Phi) = \pi_+\Big(\mathrm{d}\big(A_4(x_1, \ldots, x_3)\, \mathrm{d}x^4\big)\Big) &= \sum_{i=1}^{3} \partial_i A_4\, \pi_+(\mathrm{d}x^{14}) \\ &= \partial_1 A_4\, \rho_3^+ - \partial_2 A_4\, \rho_2^+ + \partial_3 A_4 \rho_1^+. \end{aligned}$$

Now consider the subspace $\mathbb{R}^3 \subset \mathbb{R}^4$ that is spanned by coordinates $(x_1, \ldots, x_3)$ and rename $\phi := A_4 \in C^\infty(\mathbb{R}^3, \mathfrak{g})$. It follows

$$\star_3(\mathrm{d}\phi) = \sum_{i=1}^{3} \partial_i \phi \star_3 (\mathrm{d}x^i) = \partial_1 \phi \,\mathrm{d}x^{23} - \partial_2 \phi \,\mathrm{d}x^{13} + \partial_3 \,\mathrm{d}x^{12},$$

such that $\pi_+(\mathrm{d}\Phi) = \pi_+\big(\star_3(\mathrm{d}\phi)\big)$. On the other hand, restricting the self-dual part of the gauge field strength to $\mathbb{R}^3 \subset \mathbb{R}^4$ gives the gauge field strength $F^3_{A'}$ to some connection in three dimensions. Then

$$F^+_{A'} = -\pi_+(\mathrm{d}\Phi) \iff F^3_{A'} = - \star_3 (\mathrm{d}\phi)$$

holds and determines a special gauge field in three dimensions, which depends on the Higgs field. This equation is called the **Bogomolny monopole equation**.

Another method is to consider a generalized tube of the form $Y^3 \times \mathbb{R}$, where Y is a closed three-manifold and choose a certain gauge.

Notation. Let Y be a n-dimensional closed manifold. The product $Y_+ := Y \times \mathbb{R}$ is called the ***Y*-tube** with the projection $\pi_Y : Y_+ \twoheadrightarrow Y$.

Remark 2.32. The atlas on a product manifold arises as a product of charts on Y and charts on $\mathbb{R}$. Let t denote the global coordinate induced by the canonical chart of $\mathbb{R}$. Then there exists an atlas for the Y-tube, such that any local coordinate system is of the form $(\vec{y}, t)$ and transition functions only affect the $\vec{y}$-coordinates.

Definition 2.33. Let $P \xrightarrow{\pi} Y$ be a principal G-bundle over the n-dimensional manifold Y and $\pi_Y^* P \xrightarrow{\tilde{\pi}} Y_+$ the pullback bundle on the Y-tube. Any connection $A \in \mathrm{Conn}(\pi_Y^* P)$ is locally given by

$$A = \sum_{i=1}^{n} A_i(\vec{y}, t)\,\mathrm{d}y^i + A_0(\vec{y}, t)\,\mathrm{d}t.$$

A connection is said to be in the **temporal gauge** if $A_0 = 0$, i.e. if the $\mathrm{d}t$-term vanishes for all local trivializations where the t-coordinate corresponds to the $\mathbb{R}$-direction.

The coordinate t in the remaining components can be interpreted as an additional parameter that yields an 1-parameter family of connections $A_t \in \mathrm{Conn}(P)$.

Example 2.34. Let Y be a closed three-manifold, $P \xrightarrow{\pi} Y$ a principal G-bundle, Y_+ the Y-tube and $\pi_Y^* P \xrightarrow{\tilde{\pi}} Y_+$ the corresponding pullback bundle. Furthermore, let $A \in \mathrm{Conn}(\pi_Y^* P)$ be a connection in the temporal gauge. Let A_t denote the 1-parameter family of connections on P. Using the structure equation it follows

$$F_A = \mathrm{d}A + \frac{1}{2}[A \wedge A] = \mathrm{d}A_t + \mathrm{d}t \wedge \frac{\partial A_t}{\partial t} + \frac{1}{2}[A_t \wedge A_t] = F_{A_t} - \frac{\partial A_t}{\partial t} \wedge \mathrm{d}t,$$

which relates the four-dimensional gauge field strength F_A to the 1-parameter family of three-dimensional gauge field strengths F_{A_t}. Furthermore, by the identities

$$\begin{aligned} \star_4 \,\mathrm{d}x^{ij} &= \star_3 \mathrm{d}x^{ij} \wedge \mathrm{d}t \qquad \text{for } i, j = 1, 2, 3, \\ \star_4 \,(\mathrm{d}x^i \wedge \mathrm{d}t) &= \star_3 \mathrm{d}x^i \qquad \text{for } i = 1, 2, 3 \end{aligned}$$

the 4d Hodge star is related to the 3d Hodge star. The self-dual instanton equation in four dimensions then reduce to

$$\begin{aligned} \star_4 F_A = F_A \iff \star_4 \left(F_{A_t} - \frac{\partial A_t}{\partial t} \wedge \mathrm{d}t \right) &= (\star_3 F_{A_t}) \wedge \mathrm{d}t - \star_3 \frac{\partial A_t}{\partial t} = F_{A_t} - \frac{\partial A_t}{\partial t} \wedge \mathrm{d}t \\ \Rightarrow F_{A_t} &= - \star_3 \frac{\partial A_t}{\partial t} \end{aligned}$$

which relates the gauge field strengths to the changes of the gauge field when the parameter t varies. Note that this is a second-order partial differential equation in the gauge field.

CHAPTER 3

Instantons in Higher Dimensions

The previous description of instantons in terms of connections with self-dual or anti-self-dual gauge field strength only applies to four-dimensional base spaces. There are two known methods to generalize to higher dimensions: either by complexification of a four-dimensional base manifold or by making usage of additional geometrical conditions like special holonomy. The exposition will exclusively focus on the latter approach, whose geometrical framework has been developed by the work of Salamon, Thomas, Reyez Carrión and others, see [DT98] and [Rey98].

In this chapter the basics of special holonomy are introduced and the generalization of the notion of instanton is summarized, following the exposition [Rey98] of the same title. It should be noted that the resulting instanton equations were known much earlier, e.g. [CDFN83], but not understood in terms of a unified concept.

3.1. Holonomy, curvature and Riemannian holonomy groups

Let M be a smooth manifold of dimension n and $E \xrightarrow{\pi} M$ a vector bundle. A connection on a vector bundle is usually represented by its covariant derivative $\nabla : \Gamma(E) \longrightarrow \Gamma(\mathrm{T}^*M \otimes E)$, which gives the directional derivative $\nabla_X : \Gamma(E) \longrightarrow \Gamma(E)$ for some $X \in \mathfrak{X}(M)$. Furthermore, due to $\mathfrak{gl}(k;\mathbb{R}) = \operatorname{Lie}\mathrm{GL}(k;\mathbb{R}) \cong \operatorname{End}(k;\mathbb{R})$ the adjoint Lie algebra bundle of the frame bundle of some vector bundle $E \xrightarrow{\pi} M$ is $\operatorname{ad}\operatorname{Fr}(E) = \operatorname{End}(E)$, such that the "gauge field strength" of a vector bundle is a section $F_\nabla \in \Omega^2\big(M; \operatorname{End}(E)\big)$.

A covariant derivative ∇ provides parallel transport via the following construction:

Definition 3.1. Let $\gamma : [0,1] \longrightarrow M$ be a smooth curve in the base space M. Then by virtue of the existence and uniqueness theorems of ordinary differential equations (theorems of Peano and Picard-Lindelöf) there exists an unique section $X = X(\nabla, \gamma, v) \in \Gamma(E|_{\operatorname{im}\gamma})$ such that $\nabla_{\dot\gamma(t)} X = 0$ with $X_{\gamma(0)} = v \in E_{\gamma(0)}$ holds for all $t \in [0,1]$. This gives an isomorphism

$$\begin{aligned} P^\nabla_\gamma : E_{\gamma(0)} &\xrightarrow{\cong} E_{\gamma(1)} \\ v &\mapsto X(\nabla, \gamma, v)_{\gamma(1)}, \end{aligned}$$

which is called the **parallel transport mapping**. The inverse is determined by taking the same curve γ backwards, which is usually denoted by $\gamma^{-1}(t) := \gamma(1-t)$.

Definition 3.2. Let M be a connected[a] smooth n-manifold with a covariant derivative ∇ on the vector bundle $E \xrightarrow{\pi} M$ and let $\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = \gamma(1) = x \in M$ a closed loop. The parallal mapping $P^\nabla_\gamma : E_x \xrightarrow{\cong} E_x$ is an element of $\mathrm{GL}(E_x) \cong \mathrm{GL}(n;\mathbb{R})$.

$$\operatorname{Hol}_x(\nabla) := \big\{P^\nabla_\gamma : \gamma \text{ is a closed loop with } \gamma(0) = \gamma(1) = x\big\} \subset \mathrm{GL}(E_x)$$

is called the **holonomy group based at x**, which is a Lie subgroup by the following lemma. The associated Lie algebra $\mathfrak{hol}_x(\nabla) \subseteq \operatorname{End}(E_x)$ is called the **holonomy algebra based at x**.

Lemma 3.3. $\operatorname{Hol}_x(\nabla)$ *determines a Lie subgroup of* $\mathrm{GL}(n;\mathbb{R})$ *up to conjugation and is independent of the base point* $x \in M$.

[a]If the condition of connectedness is dropped, there are points on different components, which cannot be joined by some path. Therefore each connection component has a different holonomy group.

PROOF. By attaching two curves to each other or taking a curve backwards the group multiplication and inverses in $\mathrm{Hol}_x(\nabla)$ are determined. Let $x, y \in M$ be two points in the base space connected by a smooth curve $\rho : [0,1] \longrightarrow M$ with $\rho(0) = x$ and $\rho(1) = y$. Then by attaching ρ to a closed loop at y and to ρ taken backwards it follows

$$P_\gamma \circ \mathrm{Hol}_x(\nabla) \circ P_\gamma^{-1} = \mathrm{Hol}_y(\nabla),$$

such that the holonomy group is independent of the base point up to conjugation. See [Joy00, prop. 2.2.3] for proper details. □

The holonomy group of a connection on a vector bundle (or rather its Lie algebra) is related to the curvature:

PROPOSITION 3.4. *Let M be a smooth manifold and $E \xrightarrow{\pi} M$ be a vector bundle with a covariant derivative ∇. Then the curvature $F_\nabla \in \Omega^2\big(M; \mathrm{End}(E)\big)$ at each point $x \in M$ lies in $\mathfrak{hol}_x(\nabla) \otimes \Lambda^2 \mathrm{T}_x^* M$.*

PROOF. See [Joy00, prop. 2.4.1] for the complete proof. □

Now consider Riemannian manifolds, where the metric provides a natural connection.

PROPOSITION 3.5. *Let (M, g) be a Riemannian manifold. Then there exists an uniquely determined* ***torsion-free*** *(i.e. $\nabla_U V - \nabla_V U = [U, V]_{\mathfrak{X}(M)}$ for all vector fields) connection ∇^g on $\mathrm{T}M \xrightarrow{\pi} M$ and tensors with $\nabla^g g = 0$ which is called the* ***Levi-Civita connection****.*

DEFINITION 3.6. Let (M, g) be a connected Riemannian manifold with the Levi-Civita connection ∇^g. Then $\mathrm{Hol}(g) := \mathrm{Hol}(\nabla^g)$ is called the **Riemannian holonomy group** associated to the metric g.

Since by definition the Levi-Civita connection ∇^g keeps the metric g covariantly constant, the associated holonomy group preserves the inner product at each tangent space. This implies $\mathrm{Hol}(g) \subset \mathrm{O}(n)$ for any n-dimensional connected Riemannian manifold (M, g), see [Joy00, sec. 3.1.3]. The subgroups of $\mathrm{O}(n)$ that arise as actual holonomy groups can be identified as follows:

DEFINITION 3.7. A Riemannian manifold (M, g) is called **irreducible** if it is not isomorphic to a Riemannian product $(M_1 \times M_2, g_1 \times g_2)$ of two Riemannian manifolds (M_1, g_1) and (M_2, g_2).

DEFINITION 3.8. A Riemannian manifold (M, g) is **locally symmetric** if for any point $x \in M$ there exists an open neighbourhood $U_x \subset M$ and an involutive isometry $s_x : U_x \longrightarrow U_x$ with the single fixpoint x. Conversely, (M, g) is called **nonsymmetric** if it is not locally symmetric.

REMARK 3.9. By [Joy00, thm. 3.3.8] a Riemannian manifold is locally symmetric if and only if $\nabla R = 0$ holds for the Ricci scalar curvature, i.e. the condition of a nonsymmetric space excludes spaces of constant curvature.

THEOREM 3.10 (Berger's holonomy classification). *Let (M, g) be simply-connected irreducible nonsymmetric Riemannian manifold of dimension n. Then the Riemannian holonomy group equals one of the following seven cases:*

(1) $\mathrm{Hol}(g) = \mathrm{SO}(n)$
(2) $\mathrm{Hol}(g) = \mathrm{U}(m)$ *for $n = 2m$ and $m \geq 2$ (**Kähler**)*
(3) $\mathrm{Hol}(g) = \mathrm{SU}(m)$ *for $n = 2m$ and $m \geq 2$ (**Calabi-Yau**)*
(4) $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *for $n = 4m$ and $m \geq 2$ (**Hyperkähler**)*
(5) $\mathrm{Hol}(g) = \mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$ *for $n = 4m$ and $m \geq 2$ (**quaternionic Kähler**)*
(6) $\mathrm{Hol}(g) = \mathrm{G}_2$ *for $n = 7$*
(7) $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *for $n = 8$*

This theorem was proven back in 1955 by Berger. A detailed analysis of the list and remarks concerning the proof are found in [Joy00, sec. 3.4]. Despite this early classification it took several decades until the first nontrivial compact examples were actually constructed, particularly in the two cases of exceptional holonomy G_2 and Spin(7).

3.2. Frame bundles and *G*-structures

On a manifold with special holonomy the notion of a frame bundle can be refined (or rather extended) due to the existence of certain subbundles.

DEFINITION 3.11. Let M be a smooth manifold of dimension n and $\mathrm{Fr}(M) \xrightarrow{\pi} M$ be the frame bundle with fibre $\mathrm{GL}(n;\mathbb{R})$. Furthermore, let $G \subset \mathrm{GL}(n;\mathbb{R})$ be a closed Lie subgroup. A principal G-subbundle $Q \subset \mathrm{Fr}(M)$ is called a ***G*-structure**, where the G-action is defined by restriction of the frame bundle's $\mathrm{GL}(n;\mathbb{R})$-action. A connection (i.e. horizontal distribution) $H \subset \mathrm{T}\,\mathrm{Fr}(M)$ is said to be **compatible** with the G-structure Q if the restriction $H|_{\mathrm{T}Q}$ gives a connection of Q.

Using the fundamental representation of the matrix Lie group $\mathrm{GL}(n;\mathbb{R})$ on $\mathbb{R}^n$ or its dual on $(\mathbb{R}^n)^*$, the (co-)tangent bundle can be recovered from the frame bundle by

$$\mathrm{T}M = \mathrm{Fr}(M) \times_{\mathrm{GL}} \mathbb{R}^n$$
$$\mathrm{T}^*M = \mathrm{Fr}(M) \times_{\mathrm{GL}} (\mathbb{R}^n)^*.$$

Therefore the holonomy group $\mathrm{Hol}(\nabla)$, where ∇ denotes the connection/covariant derivative on $\mathrm{T}M$, can also be attributed to the frame bundle.[b] The holonomy group is directly related to the existence of certain G-structures:

PROPOSITION 3.12. *Let M be a connected n-dimensional manifold with a connection ∇ on $\mathrm{T}M \xrightarrow{\pi} M$. Then for each Lie subgroup $G \subset \mathrm{GL}(n;\mathbb{R})$ satisfying $\mathrm{Hol}(\nabla) \subseteq G$ there exists a G-structure $Q \subset \mathrm{Fr}(M)$ which is compatible with the connection ∇.*

PROOF. See [Joy00, prop. 2.6.3] for more details. □

It should be noted that the existence of G-structures is a purely topological issue, just like the question of which groups can appear as actual holonomy groups $\mathrm{Hol}(\nabla)$ of a connection ∇ on a general bundle is answered by global topology, too. However, by requiring the G-structure to be torsion-free[c] a geometrical condition is imposed and the previous result can be restated as follows:

PROPOSITION 3.13. *Let M be a smooth n-dimensional manifold and $G \subset \mathrm{GL}(n;\mathbb{R})$ a closed Lie subgroup. Then the following two statements are equivalent:*

(1) *M admits a torsion-free G-structure $Q \subset \mathrm{Fr}(M)$.*
(2) *There exists a torsion-free connection ∇ on the tangent bundle with $\mathrm{Hol}(\nabla) \subseteq G$.*

PROOF. Again the proof in found in Joyce's book, see [Joy00, prop. 2.6.5]. □

Since the Levi-Civita connection is torsion-free by definition, any Riemannian manifold (M, g) with special holonomy possesses a natural torsion-free $\mathrm{Hol}(g)$-structure.

[b]In fact, the notion of holonomy can be defined directly for principal bundles in a similar fashion to sec. 3.1, see [Joy00, sec. 2.3] for details.

[c]The precise definition of the intrinsic torsion of a principal subbundle is rather technical. Since this property is not required in the following sections, a proper definition wil be omitted. See [Joy00, def. 2.6.4] for details.

3.3. Instanton equations in dimensions greater than four

The gauge field strength $F_A \in \Omega^2(M; \operatorname{ad} P)$ of a self-dual instanton in four dimensions corresponds to a section of the subbundle $\Lambda^2_+(M) \otimes \operatorname{ad} P$. The purpose of this section is to show a general method to identify such a distinguished subbundle if a special holonomy structure is provided.

Definition 3.14. Let $G \subset H$ be a closed Lie subgroup and let $\mathfrak{g} \subset \mathfrak{h}$ be the associated Lie algebras. Define

$$N(G) := \big\{h \in H : \mathrm{Ad}_h(G) = hGh^{-1} = G\big\}$$
$$N(\mathfrak{g}) := \big\{\xi \in \mathfrak{h} : [\xi, \zeta] \in \mathfrak{g} \text{ for all } \zeta \in \mathfrak{g}\big\} = \operatorname{Lie} N(G)$$

to be the **normalizer** of the Lie group or Lie algebra, respectively. The **centralizer**

$$C(\mathfrak{g}) := \big\{\xi \in \mathfrak{h} : [\xi, \zeta] = 0 \text{ for all } \zeta \in \mathfrak{g}\big\} \subset N(\mathfrak{g})$$

consists of those elements of $\mathfrak{h}$ that commute with the elements of $\mathfrak{g}$.

If $\mathfrak{g} \subset \mathfrak{so}(n)$ is a semi-simple Lie algebra the centralizer $C(\mathfrak{g})$ can be characterized as the orthogonal complement to $\mathfrak{g} \subset N(\mathfrak{g})$ with respect to the adjoint-invariant inner product on $N(\mathfrak{g})$, which is provided by the Killing form κ. Furthermore, if the normalizer $N(G) \subset \mathrm{SO}(n)$ is connected, the subspaces $\mathfrak{g}, C(\mathfrak{g}) \subset N(\mathfrak{g})$ are preserved by its adjoint action.

Proposition 3.15. *Let (M, g) be an oriented Riemannian manifold and $G \subset \mathrm{SO}(n)$ a closed Lie subgroup such that M admits a $N(G)$-structure $Q \subset \mathrm{Fr}(M)$ with $N(G) \subset \mathrm{SO}(n)$ being connected. Then there is a* ***distinguished subbundle*** $\tilde{\mathfrak{g}} := Q \times_{N(G)} \mathfrak{g} \subset \Lambda^2(M)$.

Proof. The details are found in the original paper [Rey98, §2]. □

Example 3.16. Consider the oriented Riemannian manifold $\mathbb{R}^4$ with the standard Euclidean metric. Note the isomorphisms $\Lambda^2_\pm \mathbb{R}^4 \cong \mathfrak{su}(2)_\pm \cong \mathfrak{so}(3)_\pm$ that follow from the identification of the Lie algebra $\mathfrak{so}(n)$ with $\Lambda^2\mathbb{R}^n$, which is given by

$$\sigma : \Lambda^2\mathbb{R}^n \xrightarrow{\cong} \mathfrak{so}(n) = \{A \in \mathrm{End}(n) \text{ skew-symmetric}\}$$
$$v \wedge w \mapsto \big[(v \wedge w)(x) := \langle v, x\rangle w - \langle w, x\rangle v\big].$$

The normalizer of $\mathrm{SU}(2)_+$ is $\mathrm{SO}(4)$, such that the $N\big(\mathrm{SU}(2)_+\big)$-structure $Q \subset \mathrm{Fr}(\mathbb{R}^4)$ is actually equal to the canonical $\mathrm{SO}(4)$-structure of the oriented Riemannian four-manifold. Therefore

$$\widetilde{\mathfrak{su}}(2)_+ = Q \times_{N(\mathrm{SU}(2)_+)} \mathfrak{su}(2)_+ = Q \times_{\mathrm{SO}(4)} \Lambda^2_+\mathbb{R}^4 = \Lambda^2_+(\mathbb{R}^4)$$

is the distinguished subbundle. Now let $P \xrightarrow{\pi} M$ be a principal G-bundle and $A \in \mathrm{Conn}(P)$ be a gauge field. The self-dual instanton equation $\star F_A = F_A$ is then equivalent to the requirement that the gauge field strength is a section of the subbundle $\widetilde{\mathfrak{su}}(2)_+ \otimes \operatorname{ad} P \subset \Lambda^2(\mathbb{R}^4) \otimes \operatorname{ad} P$.

The description of the distinguished subbundle like in the previous example can be generalized provided a $N(G)$-structure exists. As explained in the previous section, this is the case if the base manifold (M, g) has the Riemannian holonomy group $\mathrm{Hol}(g) = N(G)$. Furthermore, the distinguished subbundle induces (elliptic) complexes in a rather natural fashion:

Let $\tilde{\mathfrak{g}} := Q \times_{N(G)} \mathfrak{g} \subset \Lambda^2(M)$ be the distinguished subbundle which induces the splitting $\Lambda^2(M) = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^\perp$, where the orthogonal complement is taken with respect to the adjoint-invariant inner product induced by the Killing form by virtue of $\Lambda^2\mathbb{R}^n \cong \mathfrak{so}(n)$. This splitting is extended to other exterior powers as follows: Define vector subbundles

$$\underbrace{\begin{array}{ll} B^0 := 0 & \\ B^1 := 0 & \\ B^k := \tilde{\mathfrak{g}} \wedge \Lambda^{k-2}(M) \text{ for } k \geq 2 \qquad & A^k := (B^k)^\perp \text{ for } k \geq 0 \end{array}}_{\Lambda^k(M) = A^k \oplus B^k \text{ for all } k \geq 0}$$

group G	normalizer $N(G)$
$\mathrm{SU}(2)_+$	$\mathrm{SO}(4)$
$\mathrm{SU}(m)$	$\mathrm{U}(m)$
$\mathrm{Sp}(m)$	$\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$
G_2	G_2
$\mathrm{Spin}(7)$	$\mathrm{Spin}(7)$

TABLE 3.1. Certain Lie groups $G \subset \mathrm{SO}(n)$ whose normalizer is a Lie group appearing in Berger's classification of special holonomy groups.

as well as inclusions $i_k : B^k \hookrightarrow \Lambda^k(M)$ and projections $\pi_k : \Lambda^k(M) \twoheadrightarrow A^k$, such that one obtains a short exact sequence

$$0 \dashrightarrow B^k \xhookrightarrow{\ i_k\ } \Lambda^k(M) \xrightarrow{\ \pi_k\ } A^k \dashrightarrow 0$$

for each $k \geq 0$. With respect to the exterior derivative $\mathrm{d}_k : \Lambda^k \longrightarrow \Lambda^{k+1}$ there is the complex $\Lambda^\bullet = (\Lambda^k, \mathrm{d}_k)$ of the exterior algebra. Omitting the top and bottom zeros of the short exact sequences in the columns, one obtains

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & B^2 = \tilde{\mathfrak{g}} & \longrightarrow & B^3 & \longrightarrow & B^4 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow i_2 & & \downarrow i_3 & & \downarrow i_4 & & \\
0 & \longrightarrow & \Lambda^0 & \xrightarrow{\mathrm{d}} & \Lambda^1 & \xrightarrow{\mathrm{d}} & \Lambda^2 & \xrightarrow{\mathrm{d}} & \Lambda^3 & \xrightarrow{\mathrm{d}} & \Lambda^4 & \xrightarrow{\mathrm{d}} & \cdots \\
 & & \| & & \| & & \downarrow \pi_2 & & \downarrow \pi_3 & & \downarrow \pi_4 & & \\
0 & \longrightarrow & A^0 & \xrightarrow{D_0} & A^1 & \xrightarrow{D_1} & A^2 = \tilde{\mathfrak{g}}^\perp & \xrightarrow{D_2} & A^3 & \xrightarrow{D_3} & A^4 & \xrightarrow{D_4} & \cdots
\end{array}$$

by putting the exterior complex and the splitting together, where $D_k := \pi_{k+1} \circ \mathrm{d}_k$. The central point in the exposition of Reyes Carrión is to find conditions for (A^k, D_k) to be a complex:

PROPOSITION 3.17. *If the Levi-Civita connection has the holonomy group* $\mathrm{Hol}(g) = N(G)$ *for some group* $G \subset \mathrm{SO}(n)$ *from tab. 3.1, then* (A^k, D_k) *is a complex.*

Due to Berger's holonomy classification, all the relevant groups are found in tab. 3.1. This algebraic result together with the prescription given in ex. 3.16 motivates the following generalization of the notion of an instanton:

DEFINITION 3.18. Let (M, g) be a Riemannian manifold such that the Levi-Civita connection has the holonomy group $\mathrm{Hol}(g) = N(G)$ for some group G from tab. 3.1. The (torsion-free) $N(G)$-structure $Q \subset \mathrm{Fr}(\mathrm{T}M)$ yields the distinguished subbundle $\tilde{\mathfrak{g}} := Q \times_{N(G)} \mathfrak{g} \subset \Lambda^2(M)$. A gauge field A on the principal G-bundle $P \xrightarrow{\pi} M$ is called an **instanton** if F_A is a section of $\tilde{\mathfrak{g}} \otimes \mathrm{ad}\, P \subset \Lambda^2(M) \otimes \mathrm{ad}\, P$.

Conversely, if $\pi_{\mathfrak{g}^\perp} : \Lambda^2(M) \twoheadrightarrow \tilde{\mathfrak{g}}^\perp$ denotes the projection onto the orthogonal complement of the distinguished subbundle, then the generalized instanton equations can be stated as

$$\pi_{\tilde{\mathfrak{g}}^\perp}(F_A) = 0.$$

However, at this point the primary condition of an instanton—minimizing the Yang-Mills functional—is not necessarily met. One needs to investigate the instanton equations for each specific case seperately.

3.4. The G_2 instanton equations

From tab. 3.1 it follows that the two exceptional cases are particular simple due to the normalizers being equal to the groups, i.e. $N(\mathrm{G}_2) = \mathrm{G}_2$ and $N\big(\mathrm{Spin}(7)\big) = \mathrm{Spin}(7)$. An understanding of the corresponding instanton equation requires an appropiate understanding of the distinguished subbundles $\tilde{\mathfrak{g}}_2$ and $\widetilde{\mathfrak{spin}}(7)$ and therefore of the Lie groups G_2 and $\mathrm{Spin}(7)$ as well as their respective Lie algebras. Both groups can be constructed as stabilizers to certain k-forms, where $\mathrm{d}x^i \in \Lambda^1(\mathbb{R}^n)^* = (\mathbb{R}^n)^*$ will serve as the basis dual to $x_i \in \mathbb{R}^n$.

DEFINITION 3.19. The **exceptional group $\mathbf{G_2}$** is the stabilizer of the 3-form $\Phi_0 \in \Lambda^3(\mathbb{R}^7)^*$, that is $\mathrm{G}_2 := \{A \in \mathrm{GL}(7;\mathbb{R}) : A^*(\Phi_0) = \Phi_0\}$ where

$$\Phi_0 := \mathrm{d}x^{123} + \mathrm{d}x^{145} + \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{257} - \mathrm{d}x^{347} - \mathrm{d}x^{356}. \tag{3.1}$$

LEMMA 3.20. *G_2 is a compact 14-dimensional subgroup of $\mathrm{SO}(7)$ satisfying $\pi_i(\mathrm{G}_2) = 0$ for $i = 0,1,2$ and $\pi_3(\mathrm{G}_2) = \mathbb{Z}$.*

PROOF. Let $\iota_v\alpha$ denote the contraction mapping, i.e. the insertion of the vector v into the k-form α. Using the 3-form $\Phi_0 \in \Lambda^3(\mathbb{R}^7)^*$ from the above definition of G_2, one defines the bilinear mapping

$$\begin{aligned} b : \mathbb{R}^7 \times \mathbb{R}^7 &\longrightarrow \Lambda^7(\mathbb{R}^7)^* \\ (v,w) &\mapsto \iota_v\Phi_0 \wedge \iota_w\Phi_0 \wedge \Phi_0, \end{aligned}$$

which is G_2-equivariant from the G_2-invariance of the first two factors. Since $\iota_v\Phi_0 \in \Lambda^2(\mathbb{R}^7)^*$ is a 2-form, the mapping is symmetric under the exchange of the two arguments. With respect to the canonical basis $x_1,\dots,x_7 \in \mathbb{R}^7$ let v^i and w^i denote the components of vectors $v,w \in \mathbb{R}^7$. By some calculational effort using the definition of (3.1) it follows

$$b(v,w) = \sum_{i=1}^{7} v^i w^i \,\mathrm{d}x^{1234567} = \langle v,w\rangle \,\mathrm{d}x^{1234567},$$

where $\langle .,.\rangle$ refers to the standard Euclidean inner product on $\mathbb{R}^7$. Due to the natural relationship between the highest exterior power and the determinant mapping it follows

$$b(Av,Aw) = A^{-1} \cdot b(v,w) = \det(A^{-1})b(v,w)$$

from the G_2-equivariance, which proves

$$\langle Av,Aw\rangle = \frac{1}{\det A}\langle v,w\rangle \tag{3.2}$$

for any two $v,w \in \mathbb{R}^7$ and $A \in \mathrm{G}_2$. On the other hand, let $y_1,\dots,y_7 \in \mathbb{R}^7$ be seven vectors and let $Y := \langle y_i,y_j\rangle_{i,j=1\dots,7}$ be the 7×7-matrix of corresponding inner products. Then there is the relationship

$$\det(Y) = \Big(\underbrace{\mathrm{d}x^{1234567}(y_1 \wedge \dots \wedge y_7)}_{\in\mathbb{R}}\Big)^2. \tag{3.3}$$

Now use both identities for the particular choice of vectors $y_i := A^{-1}x_i \in \mathbb{R}^7$, which gives

$$\begin{aligned} \det Y &= \det\langle y_i,y_j\rangle_{i,j} = \det\langle A^{-1}x_i, A^{-1}x_j\rangle_{i,j} \\ &\overset{(3.2)}{=} \det\left(\frac{1}{\det(A^{-1})}\langle x_i,x_j\rangle_{i,j}\right) = \det\Big(\det(A)\langle x_i,x_j\rangle_{i,j}\Big) \\ &= \det(A)^7 \det\langle x_i,x_j\rangle_{i,j} \\ &\overset{(3.3)}{=} \det(A)^7 \mathrm{d}x^{1234567}(x_1 \wedge \dots \wedge x_7) = \det(A)^7 \end{aligned}$$

in the first case. However, using the second identity directly gives

$$\begin{aligned} \det Y &= \det\langle y_i,y_j\rangle_{i,j} \\ &\overset{(3.3)}{=} \Big(\mathrm{d}x^{1234567}(y_1 \wedge \dots \wedge y_7)\Big)^2 = \Big(\mathrm{d}x^{1234567}(A^{-1}x_1 \wedge \dots \wedge A^{-1}x_7)\Big)^2 \\ &= \Big((\det A)^{-1}\mathrm{d}x^{1234567}(x_1 \wedge \dots \wedge x_7)\Big)^2 = (\det A)^{-2} \end{aligned}$$

and this obviously yields $(\det A)^9 = 1 \in \mathbb{R}$, which leads to $\det A = 1$ for any $A \in \mathrm{G}_2$. Using identity (3.2) this gives $\langle Av, Aw\rangle = \langle v, w\rangle$ for any $A \in \mathrm{G}_2$, which establishes $G_2 \subset \mathrm{SO}(7)$. Since G_2 is obviously a closed subgroup of SO(7) it is also compact.

In order to show the remaining topological properties of G_2 the long exact homotopy sequence of a certain fibration of G_2 is used, see [Bry87, p. 540]. This proves $\pi_0(\mathrm{G}_2) = \pi_1(\mathrm{G}_2) = \pi_2(\mathrm{G}_2) = 0$ and $\pi_3(\mathrm{G}_2) \cong \mathbb{Z}$. Thus, G_2 is a compact, connected, simply-connected 14-dimensional Lie subgroup of SO(7). □

As G_2 acts irreducible on $\mathbb{R}^7$, it acts irreducible on $\Lambda^1(\mathbb{R}^7)^* = (\mathbb{R}^7)^* \cong \mathbb{R}^7$. Furthermore, $\Lambda^2(\mathbb{R}^7)^*$ contains the 14-dimensional Lie algebra $\mathfrak{g}_2$ under the identification $\mathfrak{g}_2 \subset \mathfrak{so}(7) \cong \Lambda^2\mathbb{R}^7$, which will be denoted as Λ^2_{14}. This gives the decomposition

$$\Lambda^2(\mathbb{R}^7)^* = \Lambda^2_{14} \oplus \Lambda^2_7,$$

where Λ^2_7 denotes the complement to Λ^2_{14}. Using the form $\Phi_0 \in \Lambda^3(\mathbb{R}^7)^*$ from the definition of G_2, both summands have an explicit description in terms of eigenspaces.

Definition 3.21. Let G_2 act irreducible on $\mathbb{R}^7$. On the space of k-forms define the operator

$$\begin{aligned} \star_\Phi : \Lambda^k(\mathbb{R}^7)^* &\longrightarrow \Lambda^{4-k}(\mathbb{R}^7)^* \\ \alpha &\mapsto \star_7(\Phi_0 \wedge \alpha) \end{aligned} \qquad \text{for } k = 0, \dots, 4.$$

Note that this gives an endomorphism $\star_\Phi : \Lambda^2(\mathbb{R}^7)^* \longrightarrow \Lambda^2(\mathbb{R}^7)^*$ when restricted to 2-forms.

This operator is G_2-invariant since $\Phi_0 \wedge A^*(\alpha) = A^*(\Phi_0 \wedge \alpha)$ holds for any $A \in \mathrm{G}_2$, such that $\star_\Phi$ consists of a linear combination of projections onto the G_2-irreducible summands of $\Lambda^2(\mathbb{R}^7)^*$. More precisely:

Lemma 3.22. *$\star_\Phi$ has the characteristic polynomial* $\chi(\star_\Phi; \lambda) = (\lambda+1)^{14}(\lambda-2)^7$.

Proof. This follows from a lengthy explicit calculation, i.e. writing down the operator as a $\binom{7}{2} \times \binom{7}{2} = 21 \times 21$-matrix Ξ and using the usual definition $\chi(\star_\Phi; \lambda) = \det(\Xi - \lambda\, \mathbb{1})$, which proves the statement. □

From the characteristic polynomial follows the existence of two $\star_\Phi$-eigenspaces of dimension 7 and 14, which are both G_2-irreducible by the G_2-invariance of $\star_\Phi$. Since $\Lambda^2_{14} \subset \Lambda^2(\mathbb{R}^7)^*$ is the only 14-dimensional G_2-irreducible subspace (as $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ is clearly G_2-irreducible) and Λ^2_7 its complement, this leads to the identification

$$\begin{aligned} \Lambda^2_{14} &= \mathrm{Eig}(\star_\Phi; -1) = \big\{\alpha \in \Lambda^2(\mathbb{R}^7)^* : \star_7(\Phi_0 \wedge \alpha) = -\alpha\big\}, && \text{(Lie algebra } \mathfrak{g}_2) \\ \Lambda^2_7 &= \mathrm{Eig}(\star_\Phi; 2) = \big\{\alpha \in \Lambda^2(\mathbb{R}^7)^* : \star_7(\Phi_0 \wedge \alpha) = 2\alpha\big\} && \text{(complement)} \end{aligned}$$

Due to this description of the G_2-irreducible decomposition of $\Lambda^2(\mathbb{R}^7)^*$ in terms of eigenspaces, the projector $\pi_7^{\mathrm{G}_2} : \Lambda^2(\mathbb{R}^7)^* \longrightarrow\!\!\!\!\!\rightarrow \Lambda^2_7 \subset \Lambda^2(\mathbb{R}^7)^*$ is explicitly given by

$$\pi_7^{\mathrm{G}_2} = \frac{1}{3}\,(\star_\Phi + \mathrm{Id}) = \frac{1}{3}\Big(\star_7(\Phi_0 \wedge .) + \mathrm{Id}\Big).$$

Note that $\pi_7^{\mathrm{G}_2}$ corresponds to the projector $\pi_{\tilde{\mathfrak{g}}^\perp}$ of the general instanton equation for spaces with special holonomy. The previous construction on the single vector space $\mathbb{R}^7$ lifts canonically to the tangent bundle of any G_2-manifold:

Definition 3.23. Let (M, g) be a Riemannian manifold with $\mathrm{Hol}(g) = \mathrm{G}_2$ and let $Q \subset \mathrm{Fr}(M)$ be the G_2-structure, i.e. a principal G_2-subbundle of the frame bundle. Then there exists an unique 3-form $\Phi \in \Lambda^3(M)$, such that each $\Phi_x \in \Lambda^3\mathrm{T}^*_xM$ is mapped to the prototype 3-form (3.1) by means of an oriented isomorphism, see [Joy00, p. 243] for the explicit construction. By a lapse in notation, the pair (Φ, g) will also be called a **$\mathbf{G_2}$-structure**.

DEFINITION 3.24. Let (M, g, Φ) be a G_2-manifold and $P \xrightarrow{\pi} M$ a principal G-bundle with a gauge field A. Then

$$\pi_7^{\mathrm{G}_2}(F_A) = \frac{1}{3}\left(\star_\Phi F_A + F_A\right) = \frac{1}{3}\Big(\star_7(\Phi \wedge F_A) + F_A\Big) = 0$$
$$\iff \star_7(\Phi \wedge F_A) = -F_A$$

will be referred to as the **$\mathbf{G_2}$ instanton equation**, which imply $F_A \in \Gamma(\tilde{\mathfrak{g}_2} \otimes \operatorname{ad} P)$.

LEMMA 3.25. *Let (M, g) be a 7-dimensional Riemannian manifold with $\mathrm{Hol}(g) = \mathrm{G}_2$ and let $\Phi \in \Omega^3(M)$ fix the G_2-structure. Using the G_2-irreducible decomposition $F_A = F_A^7 + F_A^{14} \in \Omega^2(M; \operatorname{ad} P)$ of the gauge field strength, the Yang-Mills functional can be reformulated as*

$$S_{\mathrm{YM}}(A) = \Xi + 3\int_M \|F_A^7\|^2 \,\mathrm{dvol},$$

for any $A \in \mathrm{Conn}(P)$, where $\Xi = \int_M \operatorname{Tr} \Phi \wedge F_A \wedge F_A$ is a topological term.

PROOF. See the proof of thm. 3.32 below, which is completely analogous. □

Therefore the definition of a G_2 instanton according to the generalization scheme presented in sec. 3.3 really fits the definition in physics, i.e. $\pi_7(F_A) = 0$ implies that $S_{\mathrm{YM}}(A)$ is minimized globally in the G_2 case.

3.5. The Spin(7) instanton equation

The outlined construction also applies to the case of Spin(7), i.e. the group is constructed as a stabilizer to a certain form, too, and a Spin(7)-irreducible decomposition of the space of 2-forms is explicitly given by the eigenspaces of a linear operator.

DEFINITION 3.26. The **exceptional holonomy group Spin(7)** is the stabilizer of the 4-form $\Omega_0 \in \Lambda^4(\mathbb{R}^8)^*$ defined by

$$\begin{aligned}\Omega_0 := &\,\mathrm{d}y^{1234} + \mathrm{d}y^{1256} + \mathrm{d}y^{1278} + \mathrm{d}y^{1357} - \mathrm{d}y^{1368} - \mathrm{d}y^{1458} - \mathrm{d}y^{1467}\\ &- \mathrm{d}y^{2358} - \mathrm{d}y^{2367} - \mathrm{d}y^{2457} + \mathrm{d}y^{2468} + \mathrm{d}y^{3456} + \mathrm{d}y^{3478} + \mathrm{d}y^{5678}\\ = &\,\mathrm{d}y^{1234} + \mathrm{d}y^{5678} + (\mathrm{d}y^{12} + \mathrm{d}y^{34}) \wedge (\mathrm{d}y^{56} + \mathrm{d}y^{78})\\ &+ (\mathrm{d}y^{13} + \mathrm{d}y^{42}) \wedge (\mathrm{d}y^{57} + \mathrm{d}y^{86})\\ &- (\mathrm{d}y^{14} + \mathrm{d}y^{23}) \wedge (\mathrm{d}y^{58} + \mathrm{d}y^{67}),\end{aligned}$$

i.e. is the subgroup $\mathrm{Spin}(7) := \{A \in \mathrm{GL}(8; \mathbb{R}) : A^*(\Omega_0) = \Omega_0\}$. Note the sign in the last row.

At first observe that the square of Ω_0 satisfies the identity $\Omega_0 \wedge \Omega_0 = 14\,\mathrm{d}y^{12345678}$. From the definition of Spin(7) it follows

$$\begin{aligned}A^*(\Omega_0 \wedge \Omega_0) &= A^*\Omega_0 \wedge A^*\Omega_0 = \Omega_0 \wedge \Omega_0\\ &= A^*(14\,\mathrm{d}y^{12345678}) = \det A \cdot 14\,\mathrm{d}y^{12345678},\end{aligned}$$

which shows $\det A = 1$ for each $A \in \mathrm{Spin}(7)$. The 4-form Ω_0 used in the definition of Spin(7) can be rewritten in two distinguished ways:

- On the one hand, there is a sort of "complex formulation" given by

$$\Omega_0 = \frac{1}{2}\alpha^2 + \mathfrak{Re}\,\beta$$
$$\text{for}\quad \begin{cases}\alpha := \mathrm{d}y^{12} + \mathrm{d}y^{34} + \mathrm{d}y^{56} + \mathrm{d}y^{78}\\ \beta := (\mathrm{d}y^1 + \mathrm{i}\,\mathrm{d}y^2) \wedge (\mathrm{d}y^3 + \mathrm{i}\,\mathrm{d}y^4) \wedge (\mathrm{d}y^5 + \mathrm{i}\,\mathrm{d}y^6) \wedge (\mathrm{d}y^7 + \mathrm{i}\,\mathrm{d}y^8).\end{cases}$$

  With respect to a complex structure J on $\mathbb{R}^8$, such that $\{\mathrm{d}y^{2i-1} + \mathrm{i}\,\mathrm{d}y^{2i}\}_{i=1,\ldots,4}$ forms a basis of $\mathbb{C}$-linear 1-forms on $\mathbb{R}^8 \cong \mathbb{C}^4$, the 4-form β can be regarded as a complex volume form.[d] Using the standard Euclidean inner product $\langle .,. \rangle$ on $\mathbb{R}^8$, the 2-form α is represented by $\alpha(v,w) = \langle Jv, w\rangle = -\langle v, Jw\rangle$ for $v, w \in \mathbb{R}^8$. This α-induced Hermitean inner product and the complex volume form β are preserved by a subgroup $H \subset \mathrm{Spin}(7)$ isomorphic to SU(4).

- On the other hand, identify the coordinates $(y_2, \ldots, y_8)$ with $(x_1, \ldots, x_7)$ as used in the definition of G_2, such that

$$\Omega_0 = \mathrm{d}y^1 \wedge \Phi_0 + \star_7 \Phi_0 \tag{3.4}$$

gives another description of the Spin(7)-defining 4-form. This splitting is stabilized by a subgroup $G \subset \mathrm{Spin}(7)$ isomorphic to G_2. Therefore this formulation of Ω_0 will be referred to as the "G_2-formulation".

A third formulation of Ω_0 will be presented in the next chapter which unveils an underlying quaternionic structure—however, the precise correspondence requires a permutation of the basis vectors in order to fix some signs. The relevant properties of the group Spin(7) are the following:

Lemma 3.27. Spin(7) *is a compact 21-dimensional Lie subgroup of* $\mathrm{SO}(8) \subset \mathrm{GL}(8;\mathbb{R})$ *satisfying* $\pi_i(\mathrm{Spin}(7)) = 0$ *for* $i = 0, 1, 2$ *and* $\pi_3(\mathrm{Spin}(7)) \cong \mathbb{Z}$. *Furthermore,* Spin(7) *is the universal twofold covering group of* SO(7), *that means* $\mathrm{Spin}(7)/\{\pm \mathrm{Id}\} \cong \mathrm{SO}(7)$.

Proof. See [Bry87, thm. 4.2] for this rather technical proof. □

Once again, the Lie algebra is contained in the space of 2-forms $\Lambda^2(\mathbb{R}^8)^*$ via the identification $\mathfrak{spin}(7) \subset \mathfrak{so}(8) \cong \Lambda^2\mathbb{R}^8$ and will be referred to as Λ^2_{21} as $\dim \mathfrak{spin}(7) = \dim \mathrm{Spin}(7) = 21$. As $\dim \Lambda^2(\mathbb{R}^8)^* = \binom{8}{2} = 28$, let $\Lambda^2_7 \subset \Lambda^2(\mathbb{R}^8)^*$ be the 7-dimensional complement of $\Lambda^2_{21} \cong \mathfrak{spin}(7)$. Again, from the 4-form Ω_0 used in the definition of the group Spin(7), one can describe the subspaces Λ^2_7 and Λ^2_{21} explicitly.

Definition 3.28. Let Spin(7) act irreducible on $\mathbb{R}^8$ and define an operator on k-form by

$$\begin{aligned} \star_\Omega : \Lambda^k(\mathbb{R}^8)^* &\longrightarrow \Lambda^{4-k}(\mathbb{R}^8)^* \\ \alpha &\longmapsto \star_8(\Omega_0 \wedge \alpha) \end{aligned} \qquad \text{for } k = 0, \ldots, 4$$

which induces an endomorphism $\star_\Omega : \Lambda^2(\mathbb{R}^8)^* \longrightarrow \Lambda^2(\mathbb{R}^8)^*$ on 2-forms.

Lemma 3.29. $\star_\Omega$ *has the characteristic polynomial* $\chi(\star_\Omega;\lambda) = (\lambda+1)^{21}(\lambda-3)^7$.

Proof. Again, the proof is just a matter of working out a lengthy calculation involving the 28×28-matrix Ξ of the operator: The Hodge star operator in local coordinates and with respect to the standard Euclidean inner product can be expressed via the Levi-Civita ϵ-symbol

$$\star_n \mathrm{d}x^{i_1\cdots i_k} = \epsilon^{i_1\cdots i_k i_{k+1}\cdots i_n}\, \mathrm{d}x^{i_{k+1}\cdots i_n}$$

for $(i_1, \ldots, i_n)$ being a permutation of $\{1, \ldots, n\}$. By the antisymmetry of the wedge product $\mathrm{d}x^{i_1\cdots i_n} = 0$ holds if any two indices $i_a = i_b$ are equal. Therefore it follows

$$\begin{aligned} \star_\Omega \mathrm{d}x^{12} &= \star(\Omega_0 \wedge \mathrm{d}x^{12}) \\ &= \star(\mathrm{d}x^{3456} \wedge \mathrm{d}x^{12} + \mathrm{d}x^{3478} \wedge \mathrm{d}x^{12} + \mathrm{d}x^{5678} \wedge \mathrm{d}x^{12}) \\ &= \star(\mathrm{d}x^{123456} + \mathrm{d}x^{123478} + \mathrm{d}x^{125678}) \\ &= \epsilon^{12345678}\, \mathrm{d}x^{78} + \epsilon^{12347856}\, \mathrm{d}x^{56} + \epsilon^{12567834}\, \mathrm{d}x^{34} \\ &= \mathrm{d}x^{78} + \mathrm{d}x^{56} + \mathrm{d}x^{34} \end{aligned}$$

and likewise for any other basis vector $\mathrm{d}x^{ij} \in \Lambda^2(\mathbb{R}^8)$ for $1 \le i < j \le 8$. The characteristic polynomial is then given by $\chi(\star_\Omega;\lambda) = \det(\Xi - \lambda\,\mathbb{1})$, cf. app. A.1. □

[d]See def. 4.6 for the precise definition of a complex structure.

By invariance of Ω_0 under the Spin(7) action, the operator is Spin(7)-invariant and thus corresponds to a linear combination of the projectors onto the Spin(7)-irreducible summands. Naturally, there is the identification

$$\Lambda^2_7 := \mathrm{Eig}(\star_\Omega; 3) = \left\{\alpha \in \Lambda^2(\mathbb{R}^8)^* : \star_8(\Omega_0 \wedge \alpha) = 3\alpha\right\},$$
$$\Lambda^2_{21} := \mathrm{Eig}(\star_\Omega; -1) = \left\{\alpha \in \Lambda^2(\mathbb{R}^8)^* : \star_8(\Omega_0 \wedge \alpha) = -\alpha\right\}$$

of the eigenspaces wth the irreducible summands and thus one arrives at the decomposition

$$\Lambda^2(\mathbb{R}^8)^* = \Lambda^2_7 \oplus \Lambda^2_{21}.$$

Once again, the projector $\pi_7^{\mathrm{Spin}(7)} : \Lambda^2(\mathbb{R}^8)^* \twoheadrightarrow \Lambda^2_7 \subset \Lambda^2(\mathbb{R}^8)^*$ is explicitly given by

$$\pi_7^{\mathrm{Spin}(7)} = \frac{1}{4}(\star_\Omega + \mathrm{Id}) = \frac{1}{4}\Big(\star_8(\Omega_0 \wedge .) + \mathrm{Id}\Big),$$

which corresponds to the projector $\pi_{\tilde{\mathfrak{g}}^\perp}$ in the Spin(7) holonomy case.

Definition 3.30. Let (M, g) be a Riemannian manifold with $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ and let $Q \subset \mathrm{Fr}(M)$ be the Spin(7)-structure. Then there exists an uniquely determinined 4-form $\Omega \in \Omega^4(M)$ such that at each point $x \in M$ the mapping $\Omega_x : \Lambda^4 \mathrm{T}_x M \longrightarrow \mathbb{R}$ is mapped to the prototype 4-form $\Omega_0 \in \Lambda^4(\mathbb{R}^8)^*$ by means of an oriented isomorphism, see [Joy00, p. 255]. The pair (Ω, g) is also called a **Spin(7)-structure**.

Definition 3.31. Let (M, g, Ω) be a Spin(7) holonomy manifold and $P \xrightarrow{\pi} M$ be a principal G-bundle with a gauge field A. Then

$$\pi_7^{\mathrm{Spin}(7)}(F_A) = \frac{1}{4}(\star_\Omega F_A + F_A) = \frac{1}{4}\Big(\star_8(\Omega \wedge F_A) + F_A\Big) = 0$$
$$\iff \star_8(\Omega \wedge F_A) = -F_A$$

will be referred to as the **Spin(7) instanton equation**.

Lemma 3.32. *Let (M, g, Ω) be a* Spin(7)*-holonomy manifold and $P \xrightarrow{\pi} M$ a principal G-bundle. Let $F_A = F_A^7 + F_A^{21} \in \Omega^2(M; \mathrm{ad}\, P)$ be the* Spin(7)*-irreducible decomposition of the gauge field strength. The Yang-Mills functional can be reformulated as*

$$S_{\mathrm{YM}}(A) = \Xi + 4 \int_M \|F_A^7\|^2 \,\mathrm{dvol}$$

for any $A \in \mathrm{Conn}(P)$, where Ξ is a topological invariant of the principal G-bundle.

Proof. From the eigenspace decomposition of $\Lambda^2(M)$ and the corresponding eigenvector equations it follows

$$\star_\Omega F_A^7 = 3F_A^7 \iff \Omega \wedge F_A^7 = 3\star F_A^7,$$
$$\star_\Omega F_A^{21} = -F_A^{21} \iff \Omega \wedge F_A^{21} = -\star F_A^{21}.$$

Consider the particular 8-form $\Omega \wedge F_A \wedge F_A$ and the decomposition $F_A = F_A^7 + F_A^{21}$, which gives

$$\begin{aligned}
\mathrm{Tr}\, \Omega \wedge F_A \wedge F_A &= \mathrm{Tr}\, \Omega \wedge (F_A^7 + F_A^{21}) \wedge (F_A^7 + F_A^{21}) \\
&= \mathrm{Tr}\Big(\Omega \wedge F_A^7 \wedge F_A^7 + \Omega \wedge F_A^7 \wedge F_A^{21} + \Omega \wedge F_A^{21} \wedge F_A^7 + \Omega \wedge F_A^{21} \wedge F_A^{21}\Big) \\
&= \mathrm{Tr}\Big(3\star F_A^7 \wedge F_A^7 + 3\star F_A^7 \wedge F_A^{21} - \star F_A^{21} \wedge F_A^7 - \star F_A^{21} \wedge F_A^{21}\Big) \\
&= \Big(3\langle F_A^7, F_A^7\rangle_{\mathrm{ad}} + 3\underbrace{\langle F_A^7, F_A^{21}\rangle_{\mathrm{ad}}}_{0} - \underbrace{\langle F_A^{21}, F_A^7\rangle_{\mathrm{ad}}}_{0} - \langle F_A^{21}, F_A^{21}\rangle_{\mathrm{ad}}\Big)\,\mathrm{dvol} \\
&= \Big(3\|F_A^7\|^2 + \|F_A^{21}\|^2\Big)\,\mathrm{dvol}.
\end{aligned}$$

Therefore it follows $\|F_A\|^2 = \|F_A^7\|^2 + \|F_A^{21}\|^2 = -\operatorname{Tr}\Omega \wedge F_A \wedge F_A + 4\|F_A^7\|^2\,\mathrm{dvol}$. Given a connection $A \in \operatorname{Conn}(P)$, the Yang-Mills functional can be rewritten as

$$S_{\mathrm{YM}}(A) = -\int_M \operatorname{Tr}\Omega \wedge F_A \wedge F_A + 4\int_M \|F_A\|^2\,\mathrm{dvol},$$

and the first term is the topological invariant Ξ by means of the Chern-Weil homomorphism. Compare [Tia00, prop. 1.3.2] for some information concerning the characteristic classes related to this topological invariant. □

Obviously, the underlying structure is equivalent in both the G_2 and Spin(7) case: The choice of either holonomy structure as a subbundle $Q \subset \operatorname{Fr}(M)$ is equivalent to a choice of a certain distinguished subbundle ϕ_Q. The 4-dimensional duality operator $\star_4$ on 2-forms is generalized to either 7d or 8d base spaces using the general scheme

$$\star_Q = \star_n(\phi_Q \wedge .),$$

where $\star_n : \Lambda^k(M) \longrightarrow \Lambda^{n-k}(M)$ denotes the Hodge operator in n dimensions. This generalizes the 4d anti-self-duality equation $\star_4 F_A = -F_A$ to $\star_Q F_A = -F_A$ in higher dimensions.

CHAPTER 4

Dimensional Reduction of Spin(7) Instantons

As shown in the case of a four-dimensional instanton, the dimensional reduction of instanton equations yields interesting links to specific solutions in lower dimensions. Using the generalization of the notion of instanton to higher-dimensional spaces with special holonomy as developed in the previous chapter, the process of dimensional reduction is applied to 8-dimensional Riemannian product manifolds with exceptional Spin(7)-holonomy.

In the first section the dimensional reduction is carried out on the product manifold $\mathbb{R}\times Z$, where Z is a G_2-manifold. In this case the Spin(7)-instanton is taken to be in the temporal gauge and it reduces to a sort of pertubated (or rather inhomogeneous) G_2-instanton on Z.

The remaining sections deal with the dimensional reduction of a translation-invariant Spin(7)-instanton on the total space of a K3 surface's trivial positive spinor bundle. First, the instanton equation and the eigenspace Λ^2_7 are reformulated in terms of quaternions and quaternion-valued differential forms. Then those constructions are specialized, such that the splitting of the 8-dimensional Spin(7)-holonomy space into two 4-dimensional space—as it is the case for each tangent space of the spinor bundle's total space—is preserved. This allows to carry out the dimensional reduction in the final section of this chapter, yielding a pair of equations resembling the 4-dimensional Seiberg-Witten equations.

4.1. Instantons on G_2-tubes in temporal gauge

Given a G_2-holonomy manifold Z, the first example of a dimensional reduction in the context of Spin(7)-instantons[a] is carried out on the product space $\mathbb{R}\times Z$, which is called a G_2-tube according to sec. 2.4 and has the properties of a Spin(7)-holonomy manifold.

LEMMA 4.1. *Let (Z, g_Z) be a 7-dimensional Riemannian manifold with* $\mathrm{Hol}(g) = \mathrm{G}_2$ *and let $\Phi \in \Omega^3(Z)$ represent the* G_2*-structure. Let $\pi_Z : \mathbb{R}\times Z \twoheadrightarrow Z$ denote the projection onto the second factor of the* G_2*-tube. On the product space $\mathbb{R}\times Z$ define a metric and 4-form by*

$$\begin{aligned} g &:= \mathrm{d}t^2 \oplus g_Z \\ \Omega &:= \mathrm{d}t \wedge \pi_Z^*\Phi + \pi_Z^*(\star_Z \Phi), \end{aligned} \tag{4.1}$$

where t is the canonical coordinate on $\mathbb{R}$. Then $(\mathbb{R}\times Z, g, \Omega)$ is an 8-dimensional Spin(7)*-holonomy manifold.*

PROOF. Using the representation of G_2- and Spin(7)-structure in terms of a differential form according to def. 3.23 and def. 3.30, the statement follows immediately from the "G_2-formulation" (3.4) of the prototype Spin(7)-form $\Omega_0 \in \Lambda^4(\mathbb{R}^8)^*$. □

The central tool in the following dimensional reduction is to express the Hodge star operator on $\mathbb{R}\times Z$ by the Hodge star on Z. Note that with respect to the coordinate t on $\mathbb{R}$ the space of 2-forms on the product manifold $\mathbb{R}\times Z$ has the form

$$\Lambda^2(\mathbb{R}\times Z) = \mathrm{d}t \wedge \pi_Z^*\Lambda^1(Z) \oplus \pi_Z^*\Lambda^2(Z)$$

such that any 2-form $\alpha \in \Omega^2(\mathbb{R}\times Z)$ can be written as $\alpha = \mathrm{d}t \wedge \pi_Z^*\alpha_1^t + \pi_Z^*\alpha_2^t$ for some 1-parameter families $\alpha_1^t \in \Omega^1(Z)$ and $\alpha_2^t \in \Omega^2(Z)$.

[a]Besides Spin(7)-instantons, there are other generalizations of the notion of instanton in eight dimensions due to complex analoga to the (real) four-dimensional case, see [DT98].

LEMMA 4.2. *In the previously described situation, the Hodge star operator* $\star$ *on* $\mathbb{R}\times Z$ *can be expressed by*

$$\star\alpha = \star\left(\mathrm{d}t\wedge\pi_Z^*\alpha_1^t + \pi_Z^*\alpha_2^t\right) = \pi_Z^*(\star_Z\alpha_1^t) + \mathrm{d}t\wedge\pi_Z^*(\star_Z\alpha_2^t),$$

where $\star_Z : \Lambda^k(Z) \xrightarrow{\cong} \Lambda^{7-k}(Z)$ *is the Hodge star operator on* Z.

PROOF. The Hodge operator is defined by the relation $\alpha\wedge\star\beta = \langle\alpha,\beta\rangle_g\,\mathrm{dvol}$ for two k-forms α,β. In the case at hand it follows $\mathrm{dvol} = \mathrm{d}t\wedge\pi_Z^*\mathrm{dvol}_Z$ from the product structure of the metric and—dropping the index t from α_1^t and α_2^t—the inner product can be rewritten as

$$\begin{aligned}\langle\alpha,\beta\rangle &= \left\langle \mathrm{d}t\wedge\widehat{\alpha_1}+\widehat{\alpha_2},\ \mathrm{d}t\wedge\widehat{\beta_1}+\widehat{\beta_2}\right\rangle_g\\ &= \left\langle \mathrm{d}t\wedge\widehat{\alpha_1},\ \mathrm{d}t\wedge\widehat{\beta_1}\right\rangle_g + \left\langle\widehat{\alpha_2},\ \widehat{\beta_2}\right\rangle_g + \underbrace{\left\langle \mathrm{d}t\wedge\widehat{\alpha_1},\ \widehat{\beta_2}\right\rangle_g}_{0} + \underbrace{\left\langle\widehat{\alpha_2},\ \mathrm{d}t\wedge\widehat{\beta_2}\right\rangle_g}_{0}\\ &= \langle\alpha_1,\beta_1\rangle_{g_Z} + \langle\alpha_2,\beta_2\rangle_{g_Z},\end{aligned}$$

where the hat "$\hat{\ }$" denotes the pullback $\pi_Z^* : \Omega^k(Z)\hookrightarrow\Omega^k(\mathbb{R}\times Z)$. By the vanishing of the square, i.e. $\mathrm{d}t^2 = \mathrm{d}t\wedge\mathrm{d}t = 0$, it follows

$$\begin{aligned}\alpha\wedge\star\beta = \langle\alpha,\beta\rangle_g\,\mathrm{dvol} &= \mathrm{d}t\wedge\pi_Z^*\left[\left(\langle\alpha_1,\beta_1\rangle_{g_Z} + \langle\alpha_2,\beta_2\rangle_{g_Z}\right)\mathrm{dvol}_Z\right]\\ &= \mathrm{d}t\wedge\pi_Z^*\left(\alpha_1\wedge\star_Z\beta_1 + \alpha_2\wedge\star_Z\beta_2\right)\\ &= \mathrm{d}t\wedge\widehat{\alpha_1}\wedge\widehat{\star_Z\beta_1} + \mathrm{d}t\wedge\widehat{\alpha_2}\wedge\widehat{\star_Z\beta_2}\\ &= (\mathrm{d}t\wedge\widehat{\alpha_1})\wedge\widehat{\star_Z\beta_1} + \widehat{\alpha_2}\wedge\left(\mathrm{d}t\wedge\widehat{\star_Z\beta_2}\right)\\ &= (\mathrm{d}t\wedge\widehat{\alpha_1}+\widehat{\alpha_2})\wedge\left(\mathrm{d}t\wedge\widehat{\star_Z\beta_2}+\widehat{\star_Z\beta_1}\right)\\ &= \alpha\wedge\left(\mathrm{d}t\wedge\widehat{\star_Z\beta_2}+\widehat{\star_Z\beta_1}\right).\end{aligned}$$

This shows $\star\beta = \star\left(\mathrm{d}t\wedge\widehat{\beta_1}+\widehat{\beta_2}\right) = \mathrm{d}t\wedge\widehat{\star_Z\beta_2}+\widehat{\star_Z\beta_1}$, which is the original statement in shorthand notation. □

THEOREM 4.3. *Let* $P\xrightarrow{\pi}M$ *be a principal* G*-bundle on the* G_2*-holonomy manifold* Z *and let* $\pi_Z^*P\xrightarrow{\tilde{\pi}}\mathbb{R}\times Z$ *denote the pullback bundle on the* G_2*-tube. Then the* Spin(7)*-instantons* $A\in\mathrm{Conn}(\pi_Z^*P)$ *in temporal gauge are described by the generalized* G_2*-instanton equation*

$$\pi_7^{G_2}(F_{A_t}) = \eta(F_{A_t}),$$

where $A_t := A|_{\{t\}\times Z}$ *is given by restriction of the gauge field to the* G_2*-slice at* t. *The perturbation* η *is given by*

$$\eta = 3\star(\star\Phi\wedge\star(\star\Phi\wedge F_{A_t})),$$

where $\star$ *is the Hodge star operator on the* G_2*-manifold* Z.

PROOF. The product metric and Spin(7)-structure $\Omega\in\Omega^4(\mathbb{R}\times Z)$ on the Z-tube are specified in lem. 4.1. Let $A\in\mathrm{Conn}(\pi_Z^*P)$ be a gauge field in the temporal gauge, cf. def. 2.33. As shown in ex. 2.34, the corresponding gauge field strength $F_A\in\Omega^2(\mathbb{R}\times Z;\mathrm{ad}\,\pi_Z^*P)$ can be expressed as

$$F_A = \pi_Z^*F_{A_t} + \mathrm{d}t\wedge\pi_Z^*\frac{\partial A_t}{\partial t} = \widehat{F_{A_t}} + \mathrm{d}t\wedge\frac{\widehat{\partial A_t}}{\partial t}, \tag{4.2}$$

where the abbreviation of the pullbacks by a hat "$\hat{\ }$" is used again. From the Spin(7)-structure (4.1), the relationship between the Hodge operators on $\mathbb{R}\times Z$ and Z as shown in lem. 4.2 and the previous expression (4.2) for the gauge field strength, it follows

$$\begin{aligned}\star_\Omega F_A = \star(\Omega\wedge F_A) &= \star\left[\left(\mathrm{d}t\wedge\widehat{\Phi}+\widehat{\star_Z\Phi}\right)\wedge\left(\widehat{F_{A_t}}+\mathrm{d}t\wedge\frac{\widehat{\partial A_t}}{\partial t}\right)\right]\\ &= \star\left[\mathrm{d}t\wedge\widehat{\Phi}\wedge\widehat{F_{A_t}}+\widehat{\star_Z\Phi}\wedge\widehat{F_{A_t}}+\widehat{\star_Z\Phi}\wedge\mathrm{d}t\wedge\frac{\widehat{\partial A_t}}{\partial t}\right]\end{aligned}$$

$$= \star\left[\mathrm{d}t \wedge \pi_Z^*\left(\Phi \wedge F_{A_t} + \star_Z \Phi \wedge \frac{\partial A_t}{\partial t}\right) + \pi_Z^*\left(\star_Z \Phi \wedge F_{A_t}\right)\right]$$
$$= \pi_Z^*\Bigg[\underbrace{\star_Z\left(\Phi \wedge F_{A_t} + \star_Z \Phi \wedge \frac{\partial A_t}{\partial t}\right)}_{\text{2-form on } Z}\Bigg] + \mathrm{d}t \wedge \pi_Z^*\Big[\underbrace{\star_Z\left(\star_Z \Phi \wedge F_{A_t}\right)}_{\text{1-form on } Z}\Big].$$

In the Spin(7) instanton equation $\pi_7^{\mathrm{Spin}(7)}(F_A) = 0 \iff \star_\Omega F_A = -F_A$ this result and (4.2) gives the two equations on 1-parameter families of differential forms on Z

$$\begin{cases} \star_Z\left(\Phi \wedge F_{A_t} + \star_Z \Phi \wedge \dfrac{\partial A_t}{\partial t}\right) = -F_{A_t} \\ \star_Z\left(\star_Z \Phi \wedge F_{A_t}\right) = -\dfrac{\partial A_t}{\partial t} \end{cases}$$
$$\iff \begin{cases} \star_Z\left(\Phi \wedge F_{A_t}\right) + F_{A_t} = -\star_Z\left(\star_Z \Phi \wedge \dfrac{\partial A_t}{\partial t}\right) \\ \star_Z\left(\star_Z \Phi \wedge F_{A_t}\right) = -\dfrac{\partial A_t}{\partial t} \end{cases}$$

Inserting the second equation into the first one and using $\pi_7^{\mathrm{G}_2} = \frac{1}{3}\left[\star_Z(\Phi \wedge .) + \mathrm{Id}\right]$ yields

$$\pi_7^{\mathrm{G}_2}(F_{A_t}) = 3 \star_Z\left(\star_Z \Phi \wedge \star_Z\left(\star_Z \Phi \wedge F_{A_t}\right)\right), \tag{4.3}$$

which gives the perturbation η on the right-hand side. □

From the "G_2-formulation" of the prototype Spin(7)-form Ω_0, see sec. 3.5 and lem. 4.1, it is not surprising that the reduction of a Spin(7)-instanton on a G_2-tube by one dimension is actually related to G_2-instantons.[b] Furthermore, the first equation (without insertion of the second one) shares a striking resemblance to the monopole equation that resulted from the dimensional reduction in ex. 2.34.

4.2. Quaternions and hyper-Kähler geometry

For the second dimensional reduction it is useful to restate the 4-form Ω_0, used in the definition of Spin(7), in terms of quaternion-valued differential forms, which is the only other division algebra over the reals besides $\mathbb{R}$ and $\mathbb{C}$.

LEMMA 4.4. *The algebra of quaternions has the following properties:*

(1) $\mathbb{H}$ *is the four-dimensional real algebra generated by the elements* e_1, e_2 *subject to the algebraic relations*

$$\{e_i, e_j\} := e_i e_j + e_j e_i = \begin{cases} 0 & : \quad i \neq j \\ -2 & : \quad i = j. \end{cases} \tag{4.4}$$

(2) $\mathbb{H}$ *is the four-dimensional real algebra generated by the elements* $\mathrm{i}, \mathrm{j}, \mathrm{k}$ *satisfying*

$$\mathrm{ij} = \mathrm{k}, \quad \mathrm{jk} = \mathrm{i}, \quad \mathrm{ki} = \mathrm{j}, \qquad \mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1 \tag{4.5}$$

(3) $\mathbb{H}$ *is a skew-field, i.e. a non-commutative field.*

(4) $\mathbb{H}$ *is a division algebra.*

(5) $\mathfrak{Re}(qp) = \mathfrak{Re}(pq)$ *for all* $p, q \in \mathbb{H}$.

PROOF. Following from the relations (4.4), $\mathbb{H} = 1\mathbb{R} \oplus e_1\mathbb{R} \oplus e_2\mathbb{R} \oplus e_1e_2\mathbb{R}$, which is a four-dimensional real vector space.[c] Define $\mathrm{i} := e_1$, $\mathrm{j} := e_2$, $\mathrm{k} := e_1e_2$, which gives the relations (4.5). This shows the equivalence of the two definitions of the quaternions $\mathbb{H}$.

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[b] Unfortunately, the author did not find a shorter expression for the term on the right side of (4.3), despite the fact that a similar equation for 1-forms is found in [FU98, §4] just below eq. (33).

[c] Note that relations (4.4) are those of a real Clifford algebra (cf. app. A.3), which shows $\mathrm{C}\ell(2) \cong \mathbb{H}$.

Let $q_1 = a_1 + b_1\mathrm{i} + c_1\mathrm{j} + d_1\mathrm{k}$ and $q_2 = a_2 + b_2\mathrm{i} + c_2\mathrm{j} + d_2\mathrm{k}$ be two quaternions. Using the relations (4.5) it follows

$$\begin{aligned} q_1 q_2 &= a_1a_2 + a_1b_2\mathrm{i} + a_1c_2\mathrm{j} + a_1d_2\mathrm{k} \\ &\quad + b_1a_2\mathrm{i} + b_1b_2\mathrm{i}^2 + b_1c_2\mathrm{ij} + b_1d_2\mathrm{ik} \\ &\quad + c_1a_2\mathrm{j} + c_1b_2\mathrm{ji} + c_1c_2\mathrm{j}^2 + c_1d_2\mathrm{jk} \\ &\quad + d_1a_2\mathrm{k} + d_1b_2\mathrm{ki} + d_1c_2\mathrm{kj} + d_1d_2\mathrm{k}^2 \\ &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) \\ &\quad + \mathrm{i}(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2) \\ &\quad + \mathrm{j}(a_1c_2 + c_1a_2 - b_1d_2 + d_1b_2) \\ &\quad + \mathrm{k}(a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2), \end{aligned}$$

which proves the last statement about the commutative real part. It also shows that the $\mathbb{H}$-multiplication is not commutative in general. The multiplicative unit element of $\mathbb{H}$ is provided by $1 = 1 + 0\mathrm{i} + 0\mathrm{j} + 0\mathrm{k} \in \mathbb{H}$, which induces a natural embedding

$$\begin{aligned} \iota_{\mathbb{R}} : \mathbb{R} &\hookrightarrow \mathbb{H} \\ a &\mapsto a + 0\mathrm{i} + 0\mathrm{j} + 0\mathrm{k} \end{aligned}$$

of the reals into the quaternions. Let $\bar{q} := a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k} \in \mathbb{H}$ denote the **quaternionic conjugate**, such that $|q|^2 := q\bar{q} = \bar{q}q$ gives the squared **norm**. It follows

$$q^{-1}|q|^2 = \bar{q} \iff q^{-1} = \frac{\bar{q}}{|q|^2},$$

which gives the **inverse** to any $0 \neq q \in \mathbb{H}$. Associativity is shown by an explicit calculation in the above fashion. Therefore, $\mathbb{H}$ is a skew-field as well as a division algebra. □

There is an equivalent description in terms of complex structures which allows to introduce "quaternionic manifolds": An almost complex structure on a $2m$-dimensional real vector space V is a $\mathbb{R}$-linear mapping $J \in \mathrm{Aut}(V)$, such that $J^2 = -\mathrm{Id}_V$ holds, i.e. it represents the (left or right) multiplication by $\mathrm{i} = \sqrt{-1}$ under the identification $V \cong \mathbb{C}^n$ as vector spaces.

Definition 4.5. Let $E \xrightarrow{\pi} M$ be a real vector bundle of rank $2m$ on a manifold M. A section of the automorphism bundle $J \in \Gamma\big(\mathrm{Aut}(E)\big) = \Gamma\big(\mathrm{GL}(E)\big)$ is called an **almost-complex structure** if $J^2 = -\mathrm{Id}_{\mathrm{T}M}$ holds, i.e. each $J_p \in \mathrm{Aut}(E_p)$ is an almost complex structure of the previous definition.

Obviously, such an almost-complex structure lifts the notion of complex multiplication with i to each vector space $E_p \cong \mathbb{R}^{2m}$. At each point $p \in M$ this allows to introduce holomorphic coordinates, but in general this is not possible in the neighbourhood of every point. Therefore, the existance of such local holomorphic coordinates requires the almost complex structure to be integrable. In case of a smooth manifold's tangent bundle, the corresponding conditions are equivalent to the vanishing of a certain tensor by means of the Newlander-Nirenberg theorem.

Definition 4.6. Let J be an almost-complex structure on $\mathrm{T}M \xrightarrow{\pi} M$ and define

$$N_J(V,W) := [V,W]_{\mathfrak{X}(M)} + J\Big([JV,W]_{\mathfrak{X}(M)} + [V,JW]_{\mathfrak{X}(M)}\Big) - [JV,JW]_{\mathfrak{X}(M)}$$

for any two vector fields $V, W \in \mathfrak{X}(M)$, which is called the **Nijenhuis tensor**. Then J is **integrable** and called a **complex structure**[d] if $N_J = 0$.

The existence of an almost-complex structure on a manifold M of dimension $2m$ is equivalent to the existence of a $\mathrm{GL}(m;\mathbb{C})$-structure. Furthermore, if the complex structure is covariantly constant with respect to the metric g of M the structure is called a **Kähler structure**.

[d]A complex structure can also be understood purely geometrical in terms of the integrability of the $\pm$i-eigenbundles to the complex structure, i.e. whether the subbundles $\mathrm{Eig}(J, \pm\mathrm{i})$ are closed under the Lie bracket.

This is equivalent to the existence of U(m)-structures on M, see [Joy00, prop. 4.4.2], such that the Riemannian metric g is Hermitean with respect to a complex structure.

DEFINITION 4.7. Let M be a smooth manifold of dimension $4m$. An **almost quaternionic structure** or **almost hyper-Kähler structure** on M is a collection of three almost-complex structures (J_1, J_2, J_3) satisfying the relations $J_1J_2J_3 = -1$ and $J_aJ_b = -J_bJ_a$ for $a \neq b$.

DEFINITION 4.8. An almost hyper-Kähler structure (J_1, J_2, J_3) on a Riemannian manifold (M, g) is called a **hyper-Kähler structure** if $\nabla^g J_i = 0$ holds for $i = 1, 2, 3$. This implies that each J_i is a complex structure.

This ensures that the metric g is compatible with the symplectic inner product induced by the hyper-Kähler structure.

REMARK 4.9. Note that there are two distinguished inequivalent (almost) hyper-Kähler structures on $\mathbb{R}^4 \cong \mathbb{H}$, which are induced by left and right $\mathbb{H}$-multiplication. Obviously, this is not the case for complex structures as $\mathbb{C}$ is commutative.

The existence of hyper-Kähler structures is equivalent to the existence of torsion-free Sp(m)-structures on M, see [Joy00, prop. 7.12]. By prop. 3.13 the existence of torsion-free Sp(m)-structures relies on $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(m)$. Therefore any hyper-Kähler manifold with holonomy group $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ possesses a natural hyper-Kähler structure, such that it can be regarded as a "quaternionic manifold".

DEFINITION 4.10. A **K3 surface** is a complex Calabi-Yau surface, i.e. a four-dimensional Riemannian manifold with holonomy $\mathrm{Hol}(g) = \mathrm{SU}(2)$.

Due to $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ each K3 surface is also a hyper-Kähler manifold. Therefore $\mathrm{T}M \xrightarrow{\pi} M$ corresponds to a $\mathbb{H}$-bundle on a K3 surface, which is equivalently described by its (torsion-free) Sp(1)-structure $Q \subset \mathrm{Fr}(M)$. As a bottom line, a K3 surface is the "quaternionic" manifold of lowest dimension that generalizes the flat quaternionic vector space $\mathbb{H}$ similar to a Riemannian surface.

4.3. Quaternionic formulation of Spin(7) eigenspaces

Differential forms with values in the quaternions are introduced similarly to complex differential forms. Let $\mathrm{d}x^1, \dots, \mathrm{d}x^{4m} \in \Lambda^1(\mathbb{R}^{4m})^*$ denote the standard basis of real 1-forms. Then define

$$\begin{aligned} \mathrm{d}q^i &:= \mathrm{d}x^{4i-3} + \mathrm{i}\,\mathrm{d}x^{4i-2} + \mathrm{j}\,\mathrm{d}x^{4i-1} + \mathrm{k}\,\mathrm{d}x^{4i} \\ \mathrm{d}\bar{q}^i &:= \mathrm{d}x^{4i-3} - \mathrm{i}\,\mathrm{d}x^{4i-2} - \mathrm{j}\,\mathrm{d}x^{4i-1} - \mathrm{k}\,\mathrm{d}x^{4i} \end{aligned} \tag{4.6}$$

for $i = 1, \dots, m$ as the prototype $\mathbb{H}$-valued 1-forms. Obviously, each $\mathrm{d}q^i$ corresponds to a projection $\pi_i : \mathbb{R}^{4m} \cong \mathbb{H}_1 \oplus \cdots \oplus \mathbb{H}_m \twoheadrightarrow \mathbb{H}_i$ onto the ith $\mathbb{H}$-summand with respect to the canonical identification $\mathbb{R}^{4m} \cong \mathbb{H}^m$. On the other hand, the $\mathrm{d}\bar{q}^i$s are the projections onto the ith $\bar{\mathbb{H}}$-summand, which has the opposite hyper-Kähler structure.

Using the antisymmetry of the wedge product and the $\mathbb{H}$-multiplication it follows

$$\begin{aligned} \mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1 = {}& \mathrm{d}x^{11} - \mathrm{i}\,\mathrm{d}x^{12} - \mathrm{j}\,\mathrm{d}x^{13} - \mathrm{k}\,\mathrm{d}x^{14} \\ & + \mathrm{i}\,\mathrm{d}x^{21} - \mathrm{i}^2\,\mathrm{d}x^{22} - \mathrm{ij}\,\mathrm{d}x^{23} - \mathrm{ik}\,\mathrm{d}x^{24} \\ & + \mathrm{j}\,\mathrm{d}x^{31} - \mathrm{ji}\,\mathrm{d}x^{32} - \mathrm{j}^2\,\mathrm{d}x^{33} - \mathrm{jk}\,\mathrm{d}x^{34} \\ & + \mathrm{k}\,\mathrm{d}x^{41} - \mathrm{ki}\,\mathrm{d}x^{42} - \mathrm{kj}\,\mathrm{d}x^{43} - \mathrm{k}^2\,\mathrm{d}x^{44} \\ = {}& -2\mathrm{i}(\mathrm{d}x^{12} + \mathrm{d}x^{34}) - 2\mathrm{j}(\mathrm{d}x^{13} + \mathrm{d}x^{42}) - 2\mathrm{k}(\mathrm{d}x^{14} + \mathrm{d}x^{23}) \\ = {}& -2(\mathrm{i}\rho_1^+ + \mathrm{j}\rho_2^+ + \mathrm{k}\rho_3^+), \end{aligned} \tag{4.7}$$

where $\rho_1^+, \rho_2^+, \rho_3^+$ correspond to the canonical basis vectors of the self-dual 2-forms on $\mathbb{R}^4$. Likewise,

$$\begin{aligned} \mathrm{d}\bar{q}^1 \wedge \mathrm{d}q^1 &= \mathrm{d}x^{11} + \mathrm{i}\, dx^{12} + \mathrm{j}\,\mathrm{d}x^{13} + \mathrm{k}\,\mathrm{d}x^{14} \\ &\quad - \mathrm{i}\,\mathrm{d}x^{21} - \mathrm{i}^2\,\mathrm{d}x^{22} - \mathrm{ij}\,\mathrm{d}x^{23} - \mathrm{ik}\,\mathrm{d}x^{24} \\ &\quad - \mathrm{j}\,\mathrm{d}x^{31} - \mathrm{ji}\,\mathrm{d}x^{32} - \mathrm{j}^2\,\mathrm{d}x^{33} - \mathrm{jk}\,\mathrm{d}x^{34} \\ &\quad - \mathrm{k}\,\mathrm{d}x^{41} - \mathrm{ki}\,\mathrm{d}x^{42} - \mathrm{kj}\,\mathrm{d}x^{43} - \mathrm{k}^2\,\mathrm{d}x^{44} \\ &= 2\mathrm{i}(\mathrm{d}x^{12} - \mathrm{d}x^{34}) + 2\mathrm{j}(\mathrm{d}x^{13} - \mathrm{d}x^{42}) + 2\mathrm{k}(\mathrm{d}x^{14} - \mathrm{d}x^{23}) \\ &= 2(\mathrm{i}\,\rho_1^- + \mathrm{j}\,\rho_2^- + \mathrm{k}\,\rho_3^-) \end{aligned} \tag{4.8}$$

produces the canonical basis $\rho_1^-, \rho_2^-, \rho_3^-$ of anti-self-dual 2-forms on $\mathbb{R}^4$. This is also true for all other $\mathbb{H}$-summands.

Now consider the special case of 8 dimensions $\mathbb{R}^8 \cong \mathbb{H}^2 = \mathbb{H}_1 \oplus \mathbb{H}_2$. Let $\rho_i^\pm$ denote the basis of (anti-)self-dual 2-forms on $\mathbb{H}_1 \cong \mathbb{R}^4$ and $\tau_i^\pm$ the basis of (anti-)self-dual 2-forms on $\mathbb{H}_2$:

$$\begin{aligned} \rho_1^\pm &= \mathrm{d}x^{12} \pm \mathrm{d}x^{34}, & \tau_1^\pm &= \mathrm{d}x^{56} \pm \mathrm{d}x^{78}, \\ \rho_2^\pm &= \mathrm{d}x^{13} \pm \mathrm{d}x^{42}, & \tau_2^\pm &= \mathrm{d}x^{57} \pm \mathrm{d}x^{86}, \\ \rho_3^\pm &= \mathrm{d}x^{14} \pm \mathrm{d}x^{23}, & \tau_3^\pm &= \mathrm{d}x^{58} \pm \mathrm{d}x^{67}. \end{aligned} \tag{4.9}$$

Note that neither the basis (4.6) of $\mathbb{H}$-valued differential forms, nor the (A)SD-bases (4.9) are normalized, which is responsible for the constant factors appearing in the final result. From the previous calculation (4.7) and the identities (2.7) it follows

$$\begin{aligned} \mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1 \wedge \mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1 &= (-2)^2(\mathrm{i}\,\rho_1^+ + \mathrm{j}\,\rho_2^+ + \mathrm{k}\,\rho_3^+) \wedge (\mathrm{i}\,\rho_1^+ + \mathrm{j}\,\rho_2^+ + \mathrm{k}\,\rho_3^+) \\ &= -4\Big[\mathrm{i}^2\,\rho_1^+ \wedge \rho_1^+ + \mathrm{ij}\,\rho_1^+ \wedge \rho_2^+ + \mathrm{ik}\,\rho_1^+ \wedge \rho_3^+ \\ &\qquad + \mathrm{ji}\,\rho_2^+ \wedge \rho_1^+ + \mathrm{j}^2\,\rho_2^+ \wedge \rho_2^+ + \mathrm{jk}\,\rho_2^+ \wedge \rho_3^+ \\ &\qquad + \mathrm{ki}\,\rho_3^+ \wedge \rho_1^+ + \mathrm{kj}\,\rho_3^+ \wedge \rho_2^+ + \mathrm{k}^2\,\rho_3^+ \wedge \rho_3^+\Big] \\ &= -4(\rho_1^+ \wedge \rho_1^+ + \rho_2^+ \wedge \rho_2^+ + \rho_3^+ \wedge \rho_3^+) \\ &= -4\sum_{i=1}^{3} \rho_i^+ \wedge \rho_i^+ \overset{(2.7)}{=} -4\sum_{i=1}^{3} \underbrace{|\rho_i^+|^2}_{2}\,\mathrm{d}x^{1234} \\ &= -4 \cdot 6\,\mathrm{d}x^{1234} \end{aligned}$$

and $\mathrm{d}q^2 \wedge \mathrm{d}\bar{q}^2 \wedge \mathrm{d}q^2 \wedge \mathrm{d}\bar{q}^2 = -4 \cdot 6\,\mathrm{d}x^{5678}$ in the same fashion. Furthermore, one obtains

$$\begin{aligned} \mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1 \wedge \mathrm{d}q^2 \wedge \mathrm{d}\bar{q}^2 &= -2 \cdot 2(\mathrm{i}\,\rho_1^+ + \mathrm{j}\,\rho_2^+ + \mathrm{k}\,\rho_3^+) \wedge (\mathrm{i}\,\tau_1^+ + \mathrm{j}\,\tau_2^+ + \mathrm{k}\,\tau_3^+) \\ &= 4\Big[\mathrm{i}^2\,\rho_1^+ \wedge \tau_1^+ + \mathrm{ij}\,\rho_1^+ \wedge \tau_2^+ + \mathrm{ik}\,\rho_1^+ \wedge \tau_3^+ \\ &\qquad + \mathrm{ji}\,\rho_2^+ \wedge \tau_1^+ + \mathrm{j}^2\,\rho_2^+ \wedge \tau_2^+ + \mathrm{jk}\,\rho_2^+ \wedge \tau_3^+ \\ &\qquad + \mathrm{ki}\,\rho_3^+ \wedge \tau_1^+ + \mathrm{kj}\,\rho_3^+ \wedge \tau_2^+ + \mathrm{k}^2\,\rho_3^+ \wedge \tau_3^+\Big] \\ &= -4\Big[\underbrace{(-\rho_1^+ \wedge \tau_1^+ - \rho_2^+ \wedge \tau_2^+ - \rho_3^+ \wedge \tau_3^+)}_{\text{real part}} + \mathrm{i}(\rho_2^+ \wedge \tau_3^+ - \rho_3^+ \wedge \tau_2^+) \\ &\qquad + \mathrm{j}(\rho_3^+ \wedge \tau_1^+ - \rho_1^+ \wedge \tau_3^+) \\ &\qquad + \mathrm{k}(\rho_1^+ \wedge \tau_2^+ - \rho_2^+ \wedge \tau_1^+)\Big] \end{aligned}$$

for the "mixed" wedge product. If $\rho^+ := (\rho_1^+, \rho_2^+, \rho_3^+)$ and $\tau^+ := (\tau_1^+, \tau_2^+, \tau_3^+)$ are two vectors of 2-forms, the last result can be restated as

$$\mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1 \wedge \mathrm{d}q^2 \wedge \mathrm{d}\bar{q}^2 = \underbrace{4\rho^+ \tilde{\bullet} \tau^+}_{\text{real part}} - \underbrace{4\kappa(\rho^+ \tilde{\times} \tau^+)}_{\text{imaginary part}},$$

where $\kappa : (b, c, d) \mapsto b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ identifies with the imaginary quaternions and "$\tilde{\bullet}$" as well as "$\tilde{\times}$" are the inner product and cross product using the wedge product as multiplication. The relationship of those products to (anti-)self-dual 2-forms on each $\mathbb{H}$-summand is used to give a quaternionic formulation of the prototype Spin(7)-form Ω_0.

Definition 4.11. Identify $\mathbb{R}^8 \cong \mathbb{H}^2$ as before and consider the real 4-form given by the real part of the previously considered products of $\mathbb{H}$-valued forms

$$\Omega_0^{\mathbb{H}} := -\frac{1}{4}\,\mathfrak{Re}\left(\frac{1}{6}\mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1 \wedge \mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1 + \frac{1}{6}\mathrm{d}q^2 \wedge \mathrm{d}\bar{q}^2 \wedge \mathrm{d}q^2 \wedge \mathrm{d}\bar{q}^2 + \mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1 \wedge \mathrm{d}q^2 \wedge \mathrm{d}\bar{q}^2\right).$$

Analogous to def. 3.26, let $\mathrm{Spin}(7)^{\mathbb{H}} := \left(A \in \mathrm{GL}(8;\mathbb{R}) : A^*\Omega_0^{\mathbb{H}} = \Omega_0^{\mathbb{H}}\right)$ be the invariance group that stabilizes $\Omega_0^{\mathbb{H}}$.

Lemma 4.12. $\mathrm{Spin}(7)^{\mathbb{H}} \cong \mathrm{Spin}(7)$.

Proof. A computation shows that $\Omega_0^{\mathbb{H}}$ and Ω_0 from def. 3.26 only differ by some signs:

$$\begin{aligned}\Omega_0^{\mathbb{H}} &= -\frac{1}{4}\left(-4\,\mathrm{d}x^{1234} - 4\,\mathrm{d}x^{5678} + 4\sum_{i=1}^{3}\rho_i^+ \wedge \tau_i^+\right)\\ &= \mathrm{d}x^{1234} + \mathrm{d}x^{5678} - \rho_1^+ \wedge \tau_1^+ - \rho_2^+ \wedge \tau_2^+ - \rho_3^+ \wedge \tau_3^+\\ \Omega_0 &= \mathrm{d}x^{1234} + \mathrm{d}x^{5678} + \rho_1^+ \wedge \tau_1^+ + \rho_2^+ \wedge \tau_2^+ - \rho_3^+ \wedge \tau_3^+ .\end{aligned}$$

Those signs are obtained by a change of coordinates. More precisely, it follows from a permutation of the coordinates $x^1 \leftrightarrow x^2$ and $x^3 \leftrightarrow x^4$ since

$$\begin{aligned}\rho_1^+ &= \mathrm{d}x^{12} + \mathrm{d}x^{34}, & & \mathrm{d}x^{21} + \mathrm{d}x^{43} = -\left(\mathrm{d}x^{12} + \mathrm{d}x^{34}\right) = -\rho_1^+,\\ \rho_2^+ &= \mathrm{d}x^{13} + \mathrm{d}x^{42}, & \xrightarrow{\text{permutation}} \quad & \mathrm{d}x^{24} + \mathrm{d}x^{31} = -\left(\mathrm{d}x^{13} + \mathrm{d}x^{42}\right) = -\rho_2^+,\\ \rho_3^+ &= \mathrm{d}x^{14} + \mathrm{d}x^{23}, & & \mathrm{d}x^{23} + \mathrm{d}x^{14} = \mathrm{d}x^{14} + \mathrm{d}x^{23} = \rho_3^+\end{aligned}$$

switches the signs as required. Such a permutation is obviously a linear mapping and therefore $\mathrm{Spin}(7)^{\mathbb{H}}$ is isomorphic to Spin(7). □

Notation. As a result of the previous lemma, it is not necessary to distinguish between both formulations. For the rest of the exposition the latter quaternionic formulation def. 4.11 will be used exclusively.

This quaternionic formulation of Spin(7) is due to Bryant and Salamon. It first appeared in their construction of complete G_2- and Spin(7)-holonomy metrics on total spaces of certain bundles over four-manifolds, see [BS89]. The central point in using quaternionic differential forms is the particular simple description of the basis for the Spin(7)-irreducible subspace $\Lambda^2_7 \subset \Lambda^2(\mathbb{R}^8)^*$.

Lemma 4.13. *With respect to the quaternionic formulation of the prototype* Spin(7)*-form* $\Omega_0 \in \Lambda^2(\mathbb{R}^8)^*$ *according to def. 4.11, define seven 2-forms* $\omega_1, \ldots, \omega_7 \in \Lambda^2(\mathbb{R}^8)^*$ *by*

$$\begin{aligned}\Lambda_3' :&\quad \frac{1}{2}\left(\mathrm{d}q^2 \wedge \mathrm{d}\bar{q}^2 - \mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1\right) =: \omega_1\mathrm{i} + \omega_2\mathrm{j} + \omega_3\mathrm{k},\\ \Lambda_4'' :&\quad \mathrm{d}\bar{q}^1 \wedge \mathrm{d}q^2 =: \omega_4 + \omega_5\mathrm{i} + \omega_6\mathrm{j} + \omega_7\mathrm{k}.\end{aligned} \tag{4.10}$$

Then $\Lambda^2_7 = \mathrm{Eig}(\star_\Omega; 3) = \mathrm{span}_{\mathbb{R}}\{\omega_1, \ldots, \omega_7\} \subset \Lambda^2(\mathbb{R}^8)^*$ *is an explicit identification of the 7-dimensional eigenspace of* $\star_\Omega$*. Furthermore, this particular basis implies a decomposition* $\Lambda^2_7 = \Lambda_3' \oplus \Lambda_4''$ *into the subspaces spanned by the basis vectors indicated above.*

Proof. The statement is shown by explicit computation of the relevant terms. From the product (4.7) of $\mathbb{H}$-valued differential forms it follows

$$\frac{1}{2}\left(\mathrm{d}q^2 \wedge \mathrm{d}\bar{q}^2 - \mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1\right) = -\left[(\tau_1^+ - \rho_1^+)\mathrm{i} + (\tau_2^+ - \rho_2^+)\mathrm{j} + (\tau_3^+ - \rho_3^+)\mathrm{k}\right],$$

such that $\omega_i = \rho_i^+ - \tau_i^+$ for $i = 1, 2, 3$. Let $\star^{(i)}$ denote the Hodge star operator on the ith summand $\mathbb{H}_i$ of $\mathbb{R}^8 = \mathbb{H}_1 \oplus \mathbb{H}_2$. In order to show that those vectors $\omega_1, \omega_2, \omega_3$ are within the subspace $\Lambda^2_7 \subset \Lambda^2(\mathbb{R}^8)^*$, consider

$$\begin{aligned}
\star_\Omega \rho_j^+ &= \star(\Omega_0 \wedge \rho_j^+) \\
&= \star\left(\mathrm{d}x^{5678} \wedge \rho_j^+ + \overbrace{\mathrm{d}x^{1234} \wedge \rho_j^+}^{0} - \sum_{i=1}^{3} \overbrace{\rho_i^+ \wedge \tau_i^+ \wedge \rho_j^+}^{2\delta_{ij}\,\mathrm{d}x^{1234} \wedge \tau_i^+}\right) \\
&= \star\left(\mathrm{d}x^{5678} \wedge \rho_j^+\right) - 2\star\left(\mathrm{d}x^{1234} \wedge \tau_j^+\right) \\
&= \star^{(1)}\rho_j^+ - 2\star^{(2)}\tau_j^+ \\
&= \rho_j^+ - 2\tau_j^+, \\
\star_\Omega \tau_j^+ &= \star \mathrm{d}x^{1234} \wedge \tau_j^+ + \underbrace{\mathrm{d}x^{5678} \wedge \tau_j^+}_{0} - \sum_{i=1}^{3} \underbrace{\rho_i^+ \wedge \tau_i^+ \wedge \tau_j^+}_{2\delta_{ij}\,\rho_i^+ \wedge \mathrm{d}x^{5678}} \\
&= \star^{(2)}\tau_j^+ - 2\star^{(1)}\rho_j^+ \\
&= \tau_j^+ - 2\rho_j^+,
\end{aligned} \tag{4.11}$$

which shows $\star_\Omega \omega_j = (\rho_j^+ - 2\tau_j^+) - (\tau_j^+ - 2\rho_j^+) = 3(\rho_j^+ - \tau_j^+) = 3\omega_j$. Therefore it follows $\omega_i \in \mathrm{Eig}(\star_\Omega; 3) = \Lambda^2_7$ for $i = 1, 2, 3$, such that

$$\begin{aligned}
\omega_1 &= \mathrm{d}x^{12} + \mathrm{d}x^{34} - \mathrm{d}x^{56} - \mathrm{d}x^{78}, \\
\omega_2 &= \mathrm{d}x^{13} + \mathrm{d}x^{42} - \mathrm{d}x^{57} - \mathrm{d}x^{86}, \\
\omega_3 &= \mathrm{d}x^{14} + \mathrm{d}x^{23} - \mathrm{d}x^{58} - \mathrm{d}x^{67}
\end{aligned} \tag{4.12}$$

are the first three basis vectors in explicit coordinates and $\Lambda'_3 := \mathrm{span}_{\mathbb{R}}\{\omega_1, \omega_2, \omega_3\}$. The remaining four vectors are obtained by an explicit computation of $\mathrm{d}\bar{q}^1 \wedge \mathrm{d}q^2$, which gives

$$\begin{aligned}
\omega_4 &= \mathrm{d}x^{15} + \mathrm{d}x^{26} + \mathrm{d}x^{37} + \mathrm{d}x^{48}, \\
\omega_5 &= -\mathrm{d}x^{25} + \mathrm{d}x^{16} + \mathrm{d}x^{47} - \mathrm{d}x^{38}, \\
\omega_6 &= -\mathrm{d}x^{35} - \mathrm{d}x^{46} + \mathrm{d}x^{17} + \mathrm{d}x^{28}, \\
\omega_7 &= -\mathrm{d}x^{45} + \mathrm{d}x^{36} - \mathrm{d}x^{27} + \mathrm{d}x^{18}.
\end{aligned} \tag{4.13}$$

According to the computation provided in app. A.1, those four vectors also satisfy $\star_\Omega \omega_i = 3\omega_i$ for $i = 4, \ldots, 7$ and define the subspace $\Lambda''_4 := \mathrm{span}_{\mathbb{R}}\{\omega_4, \ldots, \omega_7\}$. Since $\omega_1, \ldots, \omega_7$ are linearly independent, $\Lambda^2_7 = \mathrm{span}_{\mathbb{R}}\{\omega_1, \ldots, \omega_7\} \subset \Lambda^2(\mathbb{R}^8)^*$ follows. □

4.4. The projection onto the 7-dimensional eigenspace

In preparation for the description of translation-invariant Spin(7)-instantons on the total space of a (trivial) positive spinor bundle, the projection $\pi_7 : \Lambda^2(\mathbb{R}^8)^* \twoheadrightarrow \Lambda^2_7$ can be refined using the quaternionic description and the decomposition $\Lambda^2_7 = \Lambda'_3 \oplus \Lambda''_4$. Furthermore, a subgroup $H \subset \mathrm{Spin}(7)$ is identified that preserves the splitting of $\mathbb{R}^8$ into two orthogonal four-dimensional planes, e.g. $\mathbb{R}^8 = \mathbb{H} \oplus \mathbb{H}$.

LEMMA 4.14. *Let* $\mathrm{SO}(4) = \big(\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-\big)/\mathbb{Z}_2$ *act on* $\mathbb{R}^4$ *via the fundamental representation. Then the induced action on* $\Lambda^2_+\mathbb{R}^4$ *and* $\Lambda^2_-\mathbb{R}^4$ *corresponds to the adjoint action of* $\mathrm{Sp}(1)_\pm/\mathbb{Z}_2 = \mathrm{SO}(3)_\pm \subset \mathrm{SO}(4)$.

PROOF. Identify $\mathbb{R}^4 \cong \mathbb{H}$ as before and let $\ell_{[q_+,q_-]}(x) = q_+ x \bar{q}_-$ be the fundamental representation of SO(4), which induces

$$\begin{aligned}
\ell^*_{[q_+,q_-]}(\mathrm{d}q^1) &= q_+(\mathrm{d}q^1)\bar{q}_- \\
\ell^*_{[q_+,q_-]}(\mathrm{d}\bar{q}^1) &= q_-(\mathrm{d}\bar{q}^1)\bar{q}_+
\end{aligned}$$

on quaternionic 1-forms. Therefore, using the formulations (4.7) and (4.8) of the (anti-)self-dual basis, it follows

$$\begin{aligned}\ell^*_{[q_+,q_-]}(\mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1) &= q_+(\mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1)\bar{q}_+ = \mathrm{Ad}_{q_+}(\mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1)\\ \ell^*_{[q_+,q_-]}(\mathrm{d}\bar{q}^1 \wedge \mathrm{d}q^1) &= q_-(\mathrm{d}\bar{q}^1 \wedge \mathrm{d}q^1)\bar{q}_- = \mathrm{Ad}_{q_-}(\mathrm{d}\bar{q}^1 \wedge \mathrm{d}q^1).\end{aligned}$$

This gives the adjoint action of $\mathrm{Sp}(1)_\pm/\mathbb{Z}_2$ on $\mathrm{SO}(3)_\pm$. □

Remark 4.15. The left action $\ell_{[q_+,q_-]}(x) = q_+x\bar{q}_-$ of $\mathrm{SO}(4) = \big(\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-\big)/\mathbb{Z}_2$ induces a right action $r_{[q_+,q_-]}$ by acting with the inverse of $[q_+,q_-]$ from the left. Due to $q^{-1} = \bar{q}$ for all $q \in \mathrm{Sp}(1)$ it follows $[q_+,q_-]^{-1} = [\bar{q}_+,\bar{q}_-]$, such that

$$r_{[q_+,q_-]}(x) = \ell_{[q_+,q_-]^{-1}}(x) = \ell_{[\bar{q}_+,\bar{q}_-]}(x) = \bar{q}_+xq_-$$

is the corresponding right action. This relationship is important, since principal G-bundles have a right G-action.

At first the flat prototype K3 surface $\mathbb{H} \cong \mathbb{C}^2 \cong \mathbb{R}^4$ will be considered, such that $\mathbb{H}^2$ corresponds to the total space of the trivial positive spinor bundle $\not{S}^+_{\mathbb{H}} = \mathbb{H}_1 \times \mathbb{H}_2 \xrightarrow{\mathrm{pr}_1} \mathbb{H}_1$.

Theorem 4.16. *The group* $H := \big(\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_- \times \mathrm{Sp}(1)_0\big)/\mathbb{Z}_2$ *is a subgroup of* $\mathrm{Spin}(7)$ *which has a left action on* $\mathbb{R}^8 \cong \mathbb{H}^2$ *such that the following statements hold:*

(1) *The action preserves the decomposition of* $\mathbb{R}^8$ *into two 4-dimensional subspaces* $R \oplus S$.
(2) *With respect to this decomposition* $\mathbb{R}^8 = R \oplus S$, *there are two canonical embeddings of* $\mathrm{Spin}(4) \subset H$, *such that the restricted* H*-action corresponds to the fundamental action of* $\mathrm{SO}(4)$ *on the respective factor* R *or* S.
(3) *The restriction of the* H*-action to* $\mathrm{Spin}(4)_R$ *corresponds to the positive spinor representation on* S.
(4) *The* $\mathrm{Spin}(7)$*-irreducible subspace* Λ^2_7 *decomposes to* $\Lambda'_3 \oplus \Lambda''_4$ *when the induced* H*-action on 2-forms is restricted to* $\mathrm{Spin}(4)_R$, *where* $\Lambda'_3 \cong \mathfrak{sp}(1)_+$ *and* $\Lambda''_4 \cong W^-$.

Proof. • „(1) and (2)“: Identify $\mathbb{R}^8 \cong \mathbb{H}^2$ as usual and let $\mathrm{Spin}(4)_R, \mathrm{Spin}(4)_S \subset H$ be the two different embeddings of $\mathrm{Spin}(4)$ specified as follows:

$$\begin{array}{ccccc} & & H := \big(\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_- \times \mathrm{Sp}(1)_0\big)/\mathbb{Z}_2 & & \\ & \swarrow & & \searrow & \\ \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_- & & & & \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_0 \\ \| & & & & \| \\ \mathrm{Spin}(4)_R & \xleftrightarrow{\cong} & \mathrm{Spin}(4) & \xleftrightarrow{\cong} & \mathrm{Spin}(4)_S \end{array}$$

Let H act on $\mathbb{R}^8 \cong \mathbb{H}^2$, such that each of those $\mathrm{Spin}(4)$-copies acts on the respective copy of $\mathbb{H}$ via the fundamental $\mathrm{SO}(4)$-representation, i.e. for $h = [q_+,q_-,p] \in H$ define the (left) action

$$\begin{aligned}\ell_h : \mathbb{H}^2 &\longrightarrow \mathbb{H}^2\\ (x,y) &\mapsto \ell_h(x,y) := [q_+,q_-,p]\cdot(x,y) := (q_+x\bar{q}_-, q_+y\bar{p}).\end{aligned} \tag{4.14}$$

Let U denote the real 8-dimensional representation of H on $\mathbb{H}^2 \cong \mathbb{R}^8$ induced by this left action ℓ_h. Obviously, there is the decomposition of $\mathbb{R}^8$ into two four-dimensional spaces

$$U = R \oplus S \quad \text{where} \quad \begin{cases} R_{\mathbb{C}} = W^+ \otimes_{\mathbb{C}} \bar{W}^- \cong W^+ \otimes_{\mathbb{C}} W^- \\ S_{\mathbb{C}} = W^+ \otimes_{\mathbb{C}} \bar{W}^0 \cong W^+ \otimes_{\mathbb{C}} W^0. \end{cases}$$

It follows the identification $R = \mathbb{H}_1$ and $S = \mathbb{H}_2$, such that the induced action of $\mathrm{Spin}(4)$ corresponds to the fundamental $\mathrm{SO}(4)$-representation on either summand. Furthermore, on the canonical basis of quaternionic 1-forms this left H-action has the effect

$$\left.\begin{array}{r}\ell_h^*(\mathrm{d}q^1) = q_+(\mathrm{d}q^1)\bar{q}_- \\ \ell_h^*(\mathrm{d}\bar{q}^1) = q_-(\mathrm{d}\bar{q}^1)\bar{q}_+\end{array}\right\} \quad \ell_h^*(\mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1) = q_+(\mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1)\bar{q}_+$$
$$\left.\begin{array}{r}\ell_h^*(\mathrm{d}q^2) = q_+(\mathrm{d}q^2)\bar{p} \\ \ell_h^*(\mathrm{d}\bar{q}^2) = p(\mathrm{d}\bar{q}^2)\bar{q}_+\end{array}\right\} \quad \ell_h^*(\mathrm{d}q^2 \wedge \mathrm{d}\bar{q}^2) = q_+(\mathrm{d}q^2 \wedge \mathrm{d}\bar{q}^2)\bar{q}_+. \tag{4.15}$$

Therefore, each product $\mathrm{d}q^i \wedge \mathrm{d}\bar{q}^i$ transforms by the adjoint action of $\mathrm{Sp}(1)_+/\mathbb{Z}_2 \cong \mathrm{SO}(3)_+$. Since $q\bar{q} = \bar{q}q = 1$ holds for $q \in \mathrm{Sp}(1)$, it follows $\ell_h^*\Omega_0^{\mathbb{H}} = \Omega_0^{\mathbb{H}}$ by the commutativity of the real part of a $\mathbb{H}$-multiplication (cf. lem. 4.4), i.e. the Spin(7)-form is kept invariant by the action (4.14) of H. Thus, $H \subset \mathrm{Spin}(7)$ is the subgroup that preserves the decomposition of U into two four-dimensional spaces with respect to the induced Spin(4)-actions.

• „(3)“: If the action ℓ_h is restricted to the subgroup $\mathrm{Spin}(4)_R = \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$ it follows

$$\ell_{[q_+,q_-]}(x, y) = (q_+ x\bar{q}_-, q_+ y),$$

such that the restricted action on the second $\mathbb{H}$-copy S can be identified with the positive spinor representation $\Delta_4^+ \cong W^+$ of $\mathrm{Spin}(4)_R$.

• „(4)“: Now consider the explicit basis for the subspace $\Lambda_7^2 \subset \Lambda^2(\mathbb{R}^8)^*$ specified in lem. 4.13. Using (4.15) it follows

$$\begin{array}{ll}\Lambda_3' : & \ell_h^*(\mathrm{d}q^2 \wedge \mathrm{d}\bar{q}^2 - \mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1) = q_+(\mathrm{d}q^2 \wedge \mathrm{d}\bar{q}^2 - \mathrm{d}q^1 \wedge \mathrm{d}\bar{q}^1)\bar{q}_+ \\ \Lambda_4'' : & \ell_h^*(\mathrm{d}\bar{q}^1 \wedge \mathrm{d}q^2) = q_-(\mathrm{d}\bar{q}^1 \wedge \mathrm{d}q^2)\bar{p}.\end{array} \tag{4.16}$$

Besides the decomposition $\Lambda_7^2 = \Lambda_3' \oplus \Lambda_4''$ as a vector space by the particular choice of basis (4.10), the space Λ_7^2 also decomposes as a representation. Factoring the H-representation Λ_7^2 through the projection homomorphism

$$\underbrace{\big(\mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_- \times \mathrm{Sp}(1)_0\big)/\pm(1,1,1)}_{H} \longrightarrow\!\!\!\!\rightarrow \underbrace{\mathrm{Sp}(1)_+/\pm 1}_{\mathrm{SO}(3)_+} \times \underbrace{\big(\mathrm{Sp}(1)_- \times \mathrm{Sp}(1)_0\big)/\pm(1,1)}_{\mathrm{SO}(4)_{-0}}$$

yields the decomposition $(\Lambda_7^2)_{\mathbb{C}} = \mathrm{Sym}^2_{\mathbb{C}}(W^+) \oplus W^- \otimes_{\mathbb{C}} W^0$, where $\mathrm{Sym}^2 W^+ \cong \mathfrak{sp}(1)_+$ is the adjoint representation of $\mathrm{SO}(3)_+$ and $W^- \otimes_{\mathbb{C}} W^0$ is the (complexification of the) fundamental representation of $\mathrm{SO}(4)_{-0}$. It follows the identification

$$\Lambda_7^2 = \Lambda_3' \oplus \Lambda_4'' \quad \text{where} \quad \begin{cases}\Lambda_3' \cong \mathfrak{sp}(1)_+ \\ (\Lambda_4'')_{\mathbb{C}} \cong W^- \otimes_{\mathbb{C}} W^0.\end{cases} \tag{4.17}$$

The space Λ_4'' can be interpreted as a twisted negative spinor with respect to the action of $\mathrm{Spin}(4)_R = \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$ induced from ℓ_h. Therefore $\Lambda_7^2 = \Lambda_3' \oplus \Lambda_4'' \cong \mathfrak{sp}(1)_+ \oplus W^-$ follows under the restriction of the H-action to $\mathrm{Spin}(4)_R$. □

In order to construct the projection $\pi_7^{\mathrm{Spin}(7)} : \Lambda^2(\mathbb{R}^8)^* \longrightarrow\!\!\!\!\rightarrow \Lambda_7^2$ explicitly for the case at hand, the subspaces $\Lambda_3', \Lambda_4'' \subset \Lambda_7^2$ are identified as subspaces within the decomposition

$$\Lambda^2(\mathbb{R}^8)^* \cong \Lambda^2\mathbb{H}_1^* \oplus \mathbb{H}_1^* \otimes_{\mathbb{R}} \mathbb{H}_2^* \oplus \Lambda^2\mathbb{H}_2^*.$$

The projection $\pi_7^{\mathrm{Spin}(7)}$ is then constructed from π_3' and π_4''. As used in the proof of the previous theorem, the 8-dimensional real H-representation U splits to

$$U_{\mathbb{C}} = (R \oplus S)_{\mathbb{C}} \cong R_{\mathbb{C}} \oplus S_{\mathbb{C}} \cong (W^+ \otimes_{\mathbb{C}} W^-) \oplus (W^+ \otimes_{\mathbb{C}} W^0),$$

where the complexification is necessary to reveal the underlying quaternionic representations W^i of $\mathrm{Sp}(1)_i$. The canonical decomposition of the second exterior power of such a direct sum then gives

$$\Lambda^2_{\mathbb{C}}(R^*_{\mathbb{C}} \oplus S^*_{\mathbb{C}}) \cong \Lambda^2_{\mathbb{C}} R^*_{\mathbb{C}} \oplus R^*_{\mathbb{C}} \otimes_{\mathbb{C}} S^*_{\mathbb{C}} \oplus \Lambda^2_{\mathbb{C}} S^*_{\mathbb{C}},$$

where each summand on the right-hand side is given by

$$\begin{aligned}\Lambda^2_{\mathbb{C}} R^*_{\mathbb{C}} &\cong \operatorname{Sym}^2_{\mathbb{C}} W^+ \oplus \operatorname{Sym}^2_{\mathbb{C}} W^- \cong \big(\mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_-\big)_{\mathbb{C}} \\ \Lambda^2_{\mathbb{C}} S^*_{\mathbb{C}} &\cong \operatorname{Sym}^2_{\mathbb{C}} W^+ \oplus \operatorname{Sym}^2_{\mathbb{C}} W^0 \cong \big(\mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_0\big)_{\mathbb{C}} \\ R^*_{\mathbb{C}} \otimes_{\mathbb{C}} S^*_{\mathbb{C}} &\cong (\operatorname{Sym}^2_{\mathbb{C}} W^+ \otimes_{\mathbb{C}} W^- \otimes_{\mathbb{C}} W^0) \oplus (W^- \otimes_{\mathbb{C}} W^0).\end{aligned} \tag{4.18}$$

For the two "pure" summands in the decomposition the complexification can be removed, i.e.

$$\begin{aligned}\Lambda^2 R^* &\cong \mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_- \\ \Lambda^2 S^* &\cong \mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_0.\end{aligned}$$

As a result of the previous theorem $\Lambda'_3 \cong \mathfrak{sp}(1)_+$ follows as a H-representation, which shows that the projection π'_3 onto Λ'_3 is a mapping depending on the "pure" summands $\Lambda^2 R^* \oplus \Lambda^2 S^*$. Furthermore, by (4.17) it also follows $W^- \otimes_{\mathbb{C}} W^0 \cong (\Lambda''_4)_{\mathbb{C}}$ and thus the "mixed" summand is isomorphic to

$$R^*_{\mathbb{C}} \otimes_{\mathbb{C}} S^*_{\mathbb{C}} \cong \big[\big(\mathfrak{sp}(1)_+\big)_{\mathbb{C}} \otimes_{\mathbb{C}} (\Lambda''_4)_{\mathbb{C}}\big] \oplus (\Lambda''_4)_{\mathbb{C}},$$

such that the projection π''_4 is a mapping on the "mixed" summand $R^* \otimes_{\mathbb{R}} S^*$. Therefore, the projection π_7 onto the Spin(7)-irreducible subspace Λ^2_7 will be constructed from two seperate projections π'_3 and π''_4.

4.4.1. Projection onto Λ'_3. For the construction of the first projection π'_3 identify the self-dual 2-forms on R and S with the imaginary quaternions. The isomorphisms

$$\begin{aligned}\xi_R : \Lambda^2_+ R^+ &\xrightarrow{\cong} \mathfrak{Im}\,\mathbb{H} \\ \textstyle\sum_{i=1}^3 \alpha_i \rho_i^+ &\mapsto \alpha_1 \mathrm{i} + \alpha_2 \mathrm{j} + \alpha_3 \mathrm{k}\end{aligned} \qquad \text{and} \qquad \begin{aligned}\xi_S : \Lambda^2_+ S^+ &\xrightarrow{\cong} \mathfrak{Im}\,\mathbb{H} \\ \textstyle\sum_{i=1}^3 \beta_i \tau_i^+ &\mapsto \beta_1 \mathrm{i} + \beta_2 \mathrm{j} + \beta_3 \mathrm{k}\end{aligned}$$

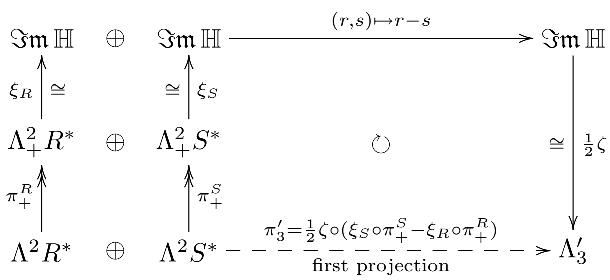
are provided by (4.7). According to lem. 4.13 the imaginary quaternions are also identified with Λ'_3 by means of

$$\begin{aligned}\zeta : \mathfrak{Im}\,\mathbb{H} &\xrightarrow{\cong} \Lambda'_3 \\ a\mathrm{i} + b\mathrm{j} + c\mathrm{k} &\mapsto a\omega_1 + b\omega_2 + c\omega_3,\end{aligned}$$

where the ω_i are the first three basis vectors of $\Lambda'_3 \subset \Lambda^2_7$ specified in (4.12). Moreover, the projections onto the self-dual 2-forms on R and S are given by

$$\begin{aligned}\pi^R_+ : \Lambda^2 R^* &\longrightarrow\!\!\!\!\!\rightarrow \Lambda^2_+ R^* \\ \alpha &\mapsto \frac{1}{2}(\star_R \alpha + \alpha)\end{aligned} \qquad \text{and} \qquad \begin{aligned}\pi^S_+ : \Lambda^2 S^* &\longrightarrow\!\!\!\!\!\rightarrow \Lambda^2_+ S^* \\ \beta &\mapsto \frac{1}{2}(\star_S \beta + \beta).\end{aligned}$$

Putting all those mappings together, the projection π'_3 is constructed as follows:



Lemma 4.17. $\pi_7^{\mathrm{Spin}(7)}|_{\Lambda^2 R^* \oplus \Lambda^2 S^*} = \pi'_3$.

Proof. This is shown by explicit computation of both sides of the equality. Let ρ_i^+, ρ_i^- be the A(SD)-basis of $\Lambda^2 \mathbb{H}_1^* \cong \Lambda^2 R^*$ and τ_i^+, τ_i^- the A(SD)-basis of $\Lambda^2 \mathbb{H}_2^* \cong \Lambda^2 S^*$, such that any element $\gamma \in \Lambda^2 R^* \oplus \Lambda^2 S^*$ is given by

$$\gamma = \sum_{i=1}^3 \Big(\alpha_i^+ \rho_i^+ + \alpha_i^- \rho_i^- + \beta_i^+ \tau_i^+ + \beta_i^- \tau_i^-\Big).$$

Analogous to (4.11) one shows $\rho_i^-, \tau_i^- \in \mathrm{Eig}(\star_\Omega; -1)$, such that $\pi_7^{\mathrm{Spin}(7)} = \frac{1}{4}(\star_\Omega + \mathrm{Id})$ has the effect

$$\pi_7^{\mathrm{Spin}(7)}(\rho_i^+) = \frac{1}{2}\omega_i \qquad \pi_7^{\mathrm{Spin}(7)}(\tau_i^+) = -\frac{1}{2}\omega_i$$
$$\pi_7^{\mathrm{Spin}(7)}(\rho_i^-) = 0 \qquad \pi_7^{\mathrm{Spin}(7)}(\tau_i^-) = 0$$

on the basis vectors. Applying the outlined construction of π_3' to the same basis elements gives

$$\begin{aligned}
\pi_3'(\rho_i^+) &= \frac{1}{2}\zeta \circ \xi_R \circ \pi_+^R(\rho_i^+) = \frac{1}{2}\zeta \circ \xi_R(\rho_i^+) = \frac{1}{2}\omega_i \\
\pi_3'(\tau_i^+) &= \frac{1}{2}\zeta \circ \big(-\xi_S \circ \pi_+^S(\tau_i^+)\big) = -\frac{1}{2}\zeta \circ \xi_S(\tau_i^+) = -\frac{1}{2}\omega_i \\
\pi_3'(\rho_i^-) &= \frac{1}{2}\zeta \circ \xi_R \circ \pi_+^R(\rho_i^-) = 0 \\
\pi_3'(\tau_i^-) &= \frac{1}{2}\zeta \circ \big(-\xi_S \circ \pi_+^S(\tau_i^-)\big) = 0.
\end{aligned}$$

This proves the statement and shows that π_3' equals $\pi_7^{\mathrm{Spin}(7)}$ on the "pure" summands. □

4.4.2. Projection onto Λ_4''. The second projection π_4'' reveals a certain structure of quaternionic homomophisms underlying the subspace $\Lambda_4'' \subset \Lambda_7^2$ as follows: Identify Λ_4'' with the quaternions via

$$\begin{aligned}
\tilde{\zeta} : \mathbb{H} &\xrightarrow{\cong} \Lambda_4'' \\
a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} &\mapsto a\omega_4 + b\omega_5 + c\omega_6 + d\omega_7
\end{aligned}$$

and use the metric on S to relate it with its dual space S^*, which will be denoted by the isomorphism $\kappa : S^* \xrightarrow{\cong} S$. Due to the canonical isomorphisms $R^* \otimes_{\mathbb{R}} S \cong \mathrm{Hom}_{\mathbb{R}}(R, S)$ and $\mathrm{Hom}_{\mathbb{H}}(R, S) \cong \mathbb{H}$, the second projection $R^* \otimes_{\mathbb{R}} S^* \twoheadrightarrow \Lambda_4''$ is actually provided by the projection $\mathrm{Hom}_{\mathbb{R}}(R, S) \twoheadrightarrow \mathrm{Hom}_{\mathbb{H}}(R, S)$.[e]

LEMMA 4.18. *Let (I_1, I_2, I_3) be a quaternionic structure on the vector space V and (J_1, J_2, J_3) be a quaternionic structure on the vector space W. Define a mapping*

$$\begin{aligned}
\mathcal{C} : \mathrm{Hom}_{\mathbb{R}}(V, W) &\longrightarrow \mathrm{Hom}_{\mathbb{R}}(V, W) \\
\Xi &\mapsto \sum_{i=1}^{3} J_i \circ \Xi \circ I_i.
\end{aligned}$$

Then the projection onto the subspace of $\mathbb{H}$-linear mappings $\mathrm{Hom}_{\mathbb{H}}(V, W)$ with respect to the quaternionic structures is given by

$$\pi_{\mathbb{H}} := \frac{1}{4}(\mathrm{Id} - \mathcal{C}) : \mathrm{Hom}_{\mathbb{R}}(V, W) \twoheadrightarrow \mathrm{Hom}_{\mathbb{H}}(V, W).$$

PROOF. Using the relations of def. 4.7 for quaternionic structures, an explicit calculation proves

$$\mathcal{C}\big(\mathcal{C}(\Xi)\big) + 2\mathcal{C}(R) - 3R = 0.$$

Since $(\lambda - 1)(\lambda + 3) = \lambda^2 + 2\lambda - 3$ the mapping $\mathcal{C}$ has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -3$. Furthermore, for $\Xi \in \mathrm{Hom}_{\mathbb{H}}(V, W)$ it follows $\Xi \circ I_i = J_i \circ \Xi_i$, such that

$$\mathcal{C}(\Xi) = \sum_{i=1}^{3} J_i \circ \Xi \circ I_i = \sum_{i=1}^{3} {J_i}^2 \circ \Xi = -3\Xi.$$

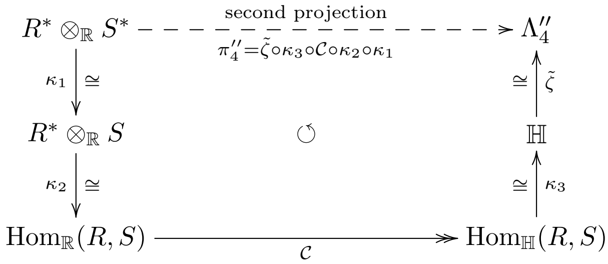
Therefore $\mathrm{Hom}_{\mathbb{H}}(V, W) = \mathrm{Eig}(\mathcal{C}; -3)$ gives a description of the space of $\mathbb{H}$-linear mappings $V \longrightarrow W$ and

$$\pi_{\mathbb{H}} := \frac{1}{\lambda_2 - \lambda_1}(\mathcal{C} - \lambda_1 \mathrm{Id}) = -\frac{1}{4}(\mathcal{C} - \mathrm{Id}) = \frac{1}{4}(\mathrm{Id} - \mathcal{C})$$

[e] My thanks to A. Haydys for some hints and thoughts on this construction. See [Hay] for some of his related work on quaternionic Dirac operators and Hyperkähler manifolds.

is the projection onto the corresponding eigenspace, which proves the statement. □

Again, putting all the aforementioned mappings together, one arrives at an explicit description of the second projection, which is shown in the following diagram:



LEMMA 4.19. $\pi_7^{\mathrm{Spin}(7)}|_{R^* \otimes_{\mathbb{R}} S^*} = \pi_4''$.

PROOF. This is shown by an explicit computation in components. With respect to coordinates $x_1, \ldots, x_4$ on R and $x_5, \ldots, x_8$ on S as before, a "mixed" 2-form is given by

$$\mu = \sum_{i=1}^{4} \sum_{j=5}^{8} \mu_{ij} \, \mathrm{d}x^i \wedge \mathrm{d}x^j \in R^* \otimes_{\mathbb{R}} S^* \subset \Lambda^2(R \oplus S)^* = \Lambda^2(\mathbb{R}^8)^*.$$

After dualizing the basis elements $\mathrm{d}x^j$ for $j = 5, \ldots, 8$ according to $\kappa : S^* \xrightarrow{\cong} S$, the 16 independent components μ_{ij} are arranged into a matrix

$$\Xi := \kappa_2 \circ \kappa_1(\mu) = \begin{pmatrix} \mu_{15} & \mu_{25} & \mu_{35} & \mu_{45} \\ \mu_{16} & \mu_{26} & \mu_{36} & \mu_{46} \\ \mu_{17} & \mu_{27} & \mu_{37} & \mu_{47} \\ \mu_{18} & \mu_{28} & \mu_{38} & \mu_{48} \end{pmatrix} \in \mathrm{Mat}_{4\times 4}(\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(R, S).$$

Let the matrices $\mathbb{1}, \ell_{\mathrm{i}}, \ell_{\mathrm{j}}, \ell_{\mathrm{k}}$ from app. A.6 represent the quaternionic structures on both R and S by left matrix multiplication. Now project this matrix $\Xi \in \mathrm{Mat}_{4\times 4}(\mathbb{R})$ onto its $\mathbb{H}$-linear part by computing

$$\begin{aligned} \pi_{\mathbb{H}} \circ \kappa_2 \circ \kappa_1(\mu) = \pi_{\mathbb{H}}(\Xi) &= \frac{1}{4}\big(\Xi - \mathcal{C}(\Xi)\big) \\ &= \frac{1}{4}\left(\Xi - \ell_{\mathrm{i}}\Xi\ell_{\mathrm{i}} - \ell_{\mathrm{j}}\Xi\ell_{\mathrm{j}} - \ell_{\mathrm{k}}\Xi\ell_{\mathrm{k}}\right) \\ &= \frac{1}{4}\big(\,\mathbb{1}(\mu_{15} + \mu_{26} + \mu_{37} + \mu_{48}) \\ &\quad + \ell_{\mathrm{i}}(\mu_{16} - \mu_{25} - \mu_{38} + \mu_{47}) \\ &\quad + \ell_{\mathrm{j}}(\mu_{17} + \mu_{28} - \mu_{35} - \mu_{46}) \\ &\quad + \ell_{\mathrm{k}}(\mu_{18} - \mu_{27} + \mu_{36} - \mu_{45})\big). \end{aligned} \tag{4.19}$$

The remaining mappings κ_3 and $\tilde{\zeta}$ just identify the quaternionic matrices with the basis vectors of Λ_4'', i.e.

$$\begin{aligned} \tilde{\zeta} \circ \kappa_3 : \mathrm{Hom}_{\mathbb{H}}(R, S) \subset \mathrm{Mat}_{4\times 4}(\mathbb{H}) &\xrightarrow{\cong} \mathbb{H} \xrightarrow{\cong} \Lambda_4'' \\ \{\mathbb{1}, \ell_{\mathrm{i}}, \ell_{\mathrm{j}}, \ell_{\mathrm{k}}\} &\mapsto \{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\} \mapsto \big\{\omega_4, \omega_5, \omega_6, \omega_7\big\}. \end{aligned}$$

Note that the indices appearing in the μ_{ij}-combinations in the final term of the computation (4.19) are equal to those of the explicit Λ_4'' basis vectors (4.13). On the other hand, the original eigenspace projection $\pi_7^{\mathrm{Spin}(7)} : \Lambda^2(\mathbb{R}^8)^* \twoheadrightarrow \Lambda^2_7$ yields

$$\begin{aligned}\pi_7^{\mathrm{Spin}(7)}(\mu) &= \frac{1}{4}\left(\star_\Omega \mu + \mu\right) \\ &= \frac{1}{4}\big(\omega_4(\mu_{15} + \mu_{26} + \mu_{37} + \mu_{48}) \\ &\quad + \omega_5(\mu_{16} - \mu_{25} - \mu_{38} + \mu_{47}) \\ &\quad + \omega_6(\mu_{17} + \mu_{28} - \mu_{35} - \mu_{46}) \\ &\quad + \omega_7(\mu_{18} - \mu_{27} + \mu_{36} - \mu_{45})\big).\end{aligned}$$

from the computations provided in app. A.1, such that both projection give the same result. □

4.4.3. Summary of the constructed projection. The outlined construction relies on the splitting $\mathbb{R}^8 = R \oplus S$, where R and S are four-dimensional fundamental representations of two different embeddings of SO(4) into H, which are induced by the restriction of the H-action. As it turns out under a suitable identification with the imaginary quaternions, the first projection π_3' is essentially the linear combination of the projections onto the self-dual 2-forms on S and R, i.e.

$$\Lambda_3' : \qquad \pi_3' : \Lambda^2 R^* \oplus \Lambda^2 S^* \twoheadrightarrow \Lambda_3' \subset \Lambda_7^2 \qquad \sim \qquad \pi_+^S - \pi_+^R.$$

Likewise, if appropriate isomorphisms are provided, the second projection π_4'' corresponds to the projection onto the $\mathbb{H}$-linear homomorphisms $R \longrightarrow S$, which can be summarized by

$$\Lambda_4'' : \qquad \pi_4'' : R^* \otimes_{\mathbb{R}} S^* \twoheadrightarrow \Lambda_4'' \subset \Lambda_7^2 \qquad \sim \qquad \pi_{\mathbb{H}}.$$

This completes the construction of an applicable formulation of the projection $\pi_7^{\mathrm{Spin}(7)}$.

4.5. Translation-invariant instantons on a K3 surface's trivial spinor bundle

For the second example of a dimensional reduction of Spin(7)-instantons, the total space of the positive spinor bundle $\not{S}_X^+$ on some K3 surface X is considered. Let $Q \subset \mathrm{Fr}(X)$ be the Sp(1)-structure of the K3 surface's Riemannian metric and $\tilde{Q}$ the associated spin structure, which is a principal Spin(4)-bundle. As shown in app. A.3 the positive and negative spinor bundle is given by

$$\not{S}_X^\pm := \tilde{Q} \times_{\Delta_4^\pm} \mathbb{H}.$$

According to the theorem of Wang [Wan89], which is summarized in [Joy00, thm. 3.6.1], the positive spinor bundle $\not{S}_X^+$ of a K3 surface is trivial, such that $\not{S}^+ \cong X \times \mathbb{H}$ follows. Therefore it is appropiate to identify the space $X \times \mathbb{R}^4 \cong X \times \mathbb{H}$ with the total space of $\not{S}_X^+$. However, $X \times \mathbb{R}^4$ is also a Spin(7)-holonomy manifold.

Lemma 4.20. *Let X be a K3 surface. On the product space $X \times \mathbb{R}^4$ there is a canonical* Spin(7)*-metric $g = g_x \oplus \mathrm{d}s^2$, where $\mathrm{d}s^2$ is the Euclidean metric on $\mathbb{R}^4$.*

Proof. The proof is similar to lem. 4.1, i.e. one shows that the Spin(7)-structure on $X \times \mathbb{R}^4$ is related to the prototype 4-form $\Omega_0 \in \Lambda^4(\mathbb{R}^8)^*$ by means of an oriented isomorphism. See [Joy00, prop. 13.1.1] for the neccessary computations. □

In fact, total spaces of bundles were the first examples of spaces with complete metrics of exceptional holonomy, see [BS89]. Using the quaternionic formulation of the Spin(7)-irreducible subspaces of the second exterior power as discussed in sec. 4.3, the dimensional reduction will be carried out.

Remark 4.21. Due to the product structure on $\not{S}_X^+ = X \times \mathbb{H}$ one can always find coordinates $x_1, \ldots, x_8$, such that $x_1, \ldots, x_4$ are local coordinates for X and $x_5, \ldots, x_8$ are global coordinates corresponding to $1, \mathrm{i}, \mathrm{j}, \mathrm{k} \in \mathbb{H}$.

Definition 4.22. Let $P \xrightarrow{\pi} X$ be a principal G-bundle and $\tilde{P} := \mathrm{pr}_1^* P \xrightarrow{\tilde{\pi}} \$_X^+ = X \times \mathbb{H}$ the associated pullback bundle over the total space of the trivial positive spinor bundle. A connection $\tilde{A} \in \mathrm{Conn}(\tilde{P})$ is called **translation-invariant** if

$$T_v^* \tilde{A} = \tilde{A} \qquad \text{holds for all } v \in \mathbb{H}, \text{ where} \qquad \begin{aligned} T_v : \$_X^+ &\xrightarrow{\cong} \$_X^+ \\ (x,\sigma) &\mapsto (x, \sigma + v) \end{aligned}$$

is the translation mapping in the direction of the fibres. The restriction of any such translation-invariant connection to a slice $X \times \{0\}$ of the total space (corresponding to the zero section of $\$_X^+ \xrightarrow{\mathrm{pr}_1} X$) gives a connection $A := \tilde{A}|_{X\times\{0\}} \in \mathrm{Conn}(P)$. Conversely, the pullback connection $A' := \mathrm{pr}_1^* A \in \mathrm{Conn}(\tilde{P})$ is translation-invariant. Then the difference

$$\Phi' := \tilde{A} - A' \in \Omega^1(\$_X^+; \mathrm{ad}\,\tilde{P})$$

is called the **Higgs field** associated to this configuration, such that

$$\tilde{A} = A' + \Phi' \in \mathrm{Conn}(\tilde{P})$$

gives a decomposition of a translation-invariant connection on $\tilde{P} \xrightarrow{\tilde{\pi}} \$_X^+$ into a pullback connection and an additional Higgs field.

Lemma 4.23. *The Higgs field $\Phi' \in \Omega^1(\$_X^+; \mathrm{ad}\,\tilde{P})$ is translation-invariant and with respect to the (local) coordinates $x_1, \ldots, x_8$ given by*

$$\Phi' = \sum_{j=5}^{8} \phi_j'(x_1, \ldots, x_4)\,\mathrm{d}x^j,$$

where the component sections $\phi_j' \in \Gamma(\tilde{P})$ are given by pullbacks of sections $\phi_j \in \Gamma(P)$.

Proof. Since $\mathrm{pr}_1(a,b) = a$ is the spinor bundle's projection mapping, any pullback via pr_1^* is independent of the coordinates on the fibres. Therefore, since $\tilde{A}$ is translation-invariant, the Higgs field $\Phi' = \tilde{A} - A' = \tilde{A} - \mathrm{pr}_1^* A$ is translation-invariant, too.

With respect to the coordinates $x_1, \ldots, x_8$ the Higgs field is given by

$$\Phi' = \sum_{j=1}^{8} \phi_j'(\underbrace{x_1, \ldots, x_4}_{\text{base space } X}, \underbrace{x_5, \ldots, x_8}_{\text{fibres } \mathbb{H}})\,\mathrm{d}x^j,$$

where $\mathrm{d}x^j \in \Omega^1(\$_X^+) = \Gamma(\mathrm{T}^*\$_X^+) = \Gamma\big(\mathrm{T}^*(X \times \mathbb{H})\big)$ is the basis of 1-forms dual to the coordinates. Due to the translation invariance of Φ' the component sections $\phi_j' \in \Gamma(\mathrm{ad}\,\tilde{P})$ must be independent of the fibre coordinates $x_5, \ldots, x_8$, such that

$$\Phi' = \sum_{j=1}^{8} \phi_j'(x_1, \ldots, x_4)\,\mathrm{d}x^j$$

follows. As the component sections $\phi_j' = \phi_j'(x_1, \ldots, x_4) \in \Gamma(\mathrm{ad}\,\tilde{P})$ only depend on the coordinates of the base space due to translation invariance, they can be represented by pullbacks of sections ϕ_j of $\mathrm{ad}\,P \xrightarrow{\pi_{\mathrm{ad}}} X$, which will be denoted by

$$\phi_j' = \mathrm{pr}_1^* \phi_j. \tag{4.20}$$

Furthermore, the restriction $\tilde{A}|_{X\times\{0\}}$ of the original connection to a single slice (i.e. the zero section) is equal to A by construction. Using $\mathrm{d}x^j|_{X\times\{0\}} = 0$ for $j = 5, \ldots, 8$ it follows

$$\Phi'|_{X\times\{0\}} = \sum_{j=1}^{4} \phi_j'(x_1, \ldots, x_4)\big|_{X\times\{0\}}\,\mathrm{d}x^j = \sum_{j=1}^{4} \phi_j(x_1, \ldots, x_4)\,\mathrm{d}x^j = 0,$$

i.e. all $\phi_j = 0$ for $j = 1, \ldots, 4$. This leaves the component sections $\phi_5, \ldots, \phi_8$ and proves the statement using (4.20). ∎

Consider the splitting $\mathrm{T}\mathcal{S}_X^+ = \mathrm{T}(X\times\mathbb{H}) = \mathrm{pr}_1^*\,\mathrm{T}X \oplus \mathrm{pr}_2^*\,\mathrm{T}\mathbb{H}$ of the spinor bundles' tangent bundle, which is equivalent to the decomposition of $\mathrm{T}\mathcal{S}_X^+$ into $\mathrm{pr}_1^*\,\mathrm{T}X$ and its orthogonal complement due to the product metric. Due to

$$\mathrm{T}\mathcal{S}_X^+\big|_{X\times\{0\}} = \mathrm{T}X \oplus \mathrm{N}X$$

and using $\mathrm{T}_p\mathbb{H} = \mathbb{H}$ for all p, it follows $\mathrm{N}X = X \times \mathbb{H} \xrightarrow{\mathrm{pr}_1} X$ for the zero sections' normal bundle. Likewise, the conormal bundle corresponds to the product bundle

$$\mathrm{N}^*X = X \times \mathbb{H}^* \xrightarrow{\mathrm{pr}_1} X$$

and if the basis of N^*X is denoted by $\mathrm{d}x^5, \ldots, \mathrm{d}x^8$ the identification with the pullback subbundle $\mathrm{pr}_1^*\,\mathrm{N}^*X \subset \mathrm{T}^*\mathcal{S}_X^+$ is obvious. From the translation invariance and the local form of the Higgs field $\Phi' \in \Omega^1(\mathcal{S}_X^+; \mathrm{ad}\,\tilde{P}) = \Gamma(\mathrm{T}^*\mathcal{S}_X^+ \otimes \mathrm{ad}\,\tilde{P})$ according to lem. 4.23 it follows $\Phi' \in \Gamma(\mathrm{pr}_1^*\,\mathrm{N}^*X \otimes \mathrm{ad}\,\tilde{P})$, which corresponds to a section of $\mathrm{N}^*X \otimes \mathrm{ad}\,P \xrightarrow{\mathrm{pr}_1 \otimes \pi_{\mathrm{ad}}} X$ by means of (4.20). This allows to regard Φ' as a "field" on the base space, i.e. a section of a bundle on the K3 surface X.

COROLLARY 4.24. *The Higgs field $\Phi' \in \Omega^1(\mathcal{S}_X^+; \mathrm{ad}\,\tilde{P})$ can be dualized to an adjoint spinor, i.e. it corresponds to a section $\Phi \in \Gamma(\mathcal{S}_X^+ \otimes \mathrm{ad}\,P)$ called the* ***adjoint Higgs spinor****.*

PROOF. Dualizing the conormal bundle $\mathrm{N}^*X \xrightarrow{\mathrm{pr}_1} X$ to the normal bundle $\mathrm{N}X \xrightarrow{\mathrm{pr}_1} X$ identifies $\mathrm{d}x^5, \ldots, \mathrm{d}x^8$ with the coordinates $x_5, \ldots, x_8$ which correspond to $1, \mathrm{i}, \mathrm{j}, \mathrm{k}$. Therefore, since the Higgs field Φ' does not depend on the fibre coordinates by the previous lemma, the dualization yields a section of $\mathcal{S}_X^+ \otimes \mathrm{ad}\,P \xrightarrow{\mathrm{pr}_1 \otimes \pi_{\mathrm{ad}}} X$, which is given by

$$\Phi := \phi_5 + \mathrm{i}\,\phi_6 + \mathrm{j}\,\phi_7 + \mathrm{k}\,\phi_8, \tag{4.21}$$

where $\phi_5, \ldots, \phi_8 \in \Gamma(\mathrm{ad}\,P)$ are the component sections with respect to the coordinates $x_5, \ldots, x_8$, which are related to $\phi_j' \in \Gamma(\mathrm{ad}\,\tilde{P})$ via the pullback relation (4.20). □

The identification of the Higgs field Φ' 1-form as an adjoint spinor Φ allows to apply the (positive part of the) Dirac operator $\not{D}_A^+ : \mathcal{S}^+ \longrightarrow \mathcal{S}^-$ to it. The dimensional reduction of the $\mathrm{Spin}(7)$ instantons in the situation of def. 4.22 is the result of the following statement.

THEOREM 4.25. *Let $\mathcal{S}_X^+ = X \times \mathbb{H}$ be the total space of the trivial positive spinor bundle over the four-dimensional K3 surface X. The subgroup $H \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8)$ that preserves the splitting of $\mathrm{T}\mathcal{S}_X^+$ in $\mathrm{pr}_1^*\,\mathrm{T}X$ and $\mathrm{pr}_2^*\,\mathrm{T}\mathbb{H}$ as well as its fibrewise action on the tangent bundle of $\mathcal{S}_X^+$ are described in thm. 4.16.*

In the situation of def. 4.22 a translation-invariant $\mathrm{Spin}(7)$-instanton on the pullback bundle $\tilde{P} = \mathrm{pr}_1^*\,P \xrightarrow{\tilde{\pi}} \mathcal{S}_X^+$ is described by the four-dimensional equations

$$\begin{cases} F_A^+ = -\dfrac{1}{4}\,[\Phi, \bar{\Phi}] \\ \not{D}_A^+\Phi = 0 \end{cases}$$

on the K3 surface X, where $\Phi \in \Gamma(\mathcal{S}_X^+ \otimes \mathrm{ad}\,P)$ is the adjoint Higgs spinor and $A' := \mathrm{pr}_1^*\,A \in \mathrm{Conn}(\tilde{P})$ the pullback of a connection $A \in \mathrm{Conn}(P)$.

PROOF. From the previously described splitting $\tilde{A} = A' + \Phi'$ of a translation-invariant connection on $\tilde{P} = \mathrm{pr}_1^*\,P \xrightarrow{\tilde{\pi}} \mathcal{S}_X^+$, the gauge field strength

$$F_{\tilde{A}} = F_{A'+\Phi'} = F_{A'} + \mathrm{d}_{A'}\Phi' + \frac{1}{2}[\Phi' \wedge \Phi']$$

decomposes according to the structure equation. Following the split construction $\pi_7 = \pi_3' \oplus \pi_4''$ of the projection onto the 7-dimensional $\mathrm{Spin}(7)$-irreducible subspace Λ_7^2, the $\mathrm{Spin}(7)$-instanton equation $\pi_7(F_{\tilde{A}}) = 0$ decomposes to

$$\begin{cases} \pi_3'\left(F_{A'} + \frac{1}{2}[\Phi' \wedge \Phi']\right) = 0 \\ \pi_4''\left(\mathrm{d}_{A'}\Phi'\right) = 0. \end{cases} \tag{4.22}$$

The first of the two equations can be simplified as follows: Consider the Lie bracket of the adjoint Higgs spinor (4.21) with its quaternionic conjugate

$$\begin{aligned}\left[\Phi,\bar{\Phi}\right] &= [\phi_5+\mathrm{i}\,\phi_6+\mathrm{j}\,\phi_7+\mathrm{k}\,\phi_8,\phi_5-\mathrm{i}\,\phi_6-\mathrm{j}\,\phi_7-\mathrm{k}\,\phi_8]\\ &= -2\Big(\big([\phi_5,\phi_6]+[\phi_7,\phi_8]\big)\mathrm{i}\\ &\quad+\big([\phi_5,\phi_7]+[\phi_8,\phi_6]\big)\mathrm{j}\\ &\quad+\big([\phi_5,\phi_8]+[\phi_6,\phi_7]\big)\mathrm{k}\Big).\end{aligned}$$

By definition of the projection π_3', only the self-dual part of $[\Phi'\wedge\Phi']$ is relevant. Due to

$$\begin{aligned}\pi_S^+\left([\Phi'\wedge\Phi']\right) &= \pi_S^+\left(2\sum_{i<j}^{5\ldots8}[\phi_i',\phi_j']\otimes \mathrm{d}x^i\wedge\mathrm{d}x^j\right)\\ &= \Big(\big([\phi_5',\phi_6']+[\phi_7',\phi_8']\big)\tau_1^+\\ &\quad+\big([\phi_5',\phi_7']+[\phi_8',\phi_6']\big)\tau_2^+\\ &\quad+\big([\phi_5',\phi_8']+[\phi_6',\phi_7']\big)\tau_3^+\Big),\end{aligned}$$

where τ_i^+ is the self-dual basis from (4.9), the identification $\xi_S:\{\tau_1^+,\tau_2^+,\tau_3^+\}\mapsto\{\mathrm{i},\mathrm{j},\mathrm{k}\}$ with the imaginary quaterions gives

$$\xi_S\circ\pi_S^+\big([\Phi'\wedge\Phi']\big) = -\frac{1}{2}\,\mathrm{pr}_1^*\left[\Phi,\bar{\Phi}\right].$$

Together with the pullback gauge field strength $F_{A'}=F_{\mathrm{pr}_1^*A}=\mathrm{pr}_1^*F_A\in\Gamma(\mathrm{pr}_1^*\Lambda^2\mathrm{T}^*X\otimes\mathrm{ad}\,\tilde{P})$, the first equation of (4.22) yields

$$\begin{aligned}\pi_3'\left(F_{A'}+\frac{1}{2}[\Phi'\wedge\Phi']\right) &= \zeta\circ\Big(\xi_S\circ\pi_S^+\left(\tfrac{1}{2}[\Phi'\wedge\Phi']\right)-\xi_R\circ\pi_R^+(\mathrm{pr}_1^*F_A)\Big)\\ &= -\zeta\circ\left(\frac{1}{4}\,\mathrm{pr}_1^*\left[\Phi,\bar{\Phi}\right]+\xi_R(\mathrm{pr}_1^*F_A^+)\right)=0\\ \iff\ \xi_R(\mathrm{pr}_1^*F_A^+) &= -\frac{1}{4}\,\mathrm{pr}_1^*\left[\Phi,\bar{\Phi}\right],\end{aligned}$$

where ξ_R and ζ are the bundle extensions of the identifications of with $\Im\mathfrak{m}\,\mathbb{H}$ and Λ_3' from the previous section. If the identification ξ_R with the imaginary quaternions and the pullback are suppressed, this yields one half of the statement, i.e. one obtains

$$F_A^+ = -\frac{1}{4}\left[\Phi,\bar{\Phi}\right],$$

which relates the self-dual part of the four-dimensional gauge field strength to the adjoint Higgs spinor.

In order to evaluate the second equation $\pi_4''(\mathrm{d}_{A'}\Phi')=0$ one follows a similar procedure: Let ∇_i denote the covariant derivative on the adjoint pullback bundle $\mathrm{ad}\,\tilde{P}\xrightarrow{\tilde{\pi}_{\mathrm{ad}}}\$_X^+$ in the direction of the coordinate x_i and identify the gradient operator with the quaternions, i.e. define

$$\nabla^{\mathbb{H}} := \nabla_1+\mathrm{i}\nabla_2+\mathrm{j}\nabla_3+\mathrm{k}\nabla_4.$$

Then consider the "conjugate gradient" $\bar{\nabla}^{\mathbb{H}}$ of the adjoint Higgs spinor (4.21), which gives

$$\begin{aligned}\bar{\nabla}^{\mathbb{H}}\Phi &= (\nabla_1-\mathrm{i}\nabla_2-\mathrm{j}\nabla_3-\mathrm{k}\nabla_4)\,(\phi_5+\mathrm{i}\phi_6+\mathrm{j}\phi_7+\mathrm{k}\phi_8)\\ &= (\nabla_1\phi_5+\nabla_2\phi_6+\nabla_3\phi_7+\nabla_4\phi_8)\\ &\quad+\mathrm{i}\,(-\nabla_2\phi_5+\nabla_1\phi_6+\nabla_4\phi_7-\nabla_3\phi_8)\\ &\quad+\mathrm{j}\,(-\nabla_3\phi_5-\nabla_4\phi_6+\nabla_1\phi_7+\nabla_2\phi_8)\\ &\quad+\mathrm{k}\,(-\nabla_4\phi_5+\nabla_3\phi_6-\nabla_2\phi_7+\nabla_1\phi_8)\end{aligned}\tag{4.23}$$

On the other hand, using the local description of Φ' and local coordinates, the covariant exterior derivative $\mathrm{d}_{A'}\Phi'$ is explicitly given by

$$\mathrm{d}_{A'}\Phi' = \mathrm{d}_{A'}\left(\sum_{j=5}^{8}\phi'_j(x_1,\dots,x_4)\,\mathrm{d}x^j\right) = \sum_{j=5}^{8}\sum_{i=1}^{4}\left(\nabla_i^{A'}\phi'_j\right)\,\mathrm{d}x^i\wedge\mathrm{d}x^j.$$

After dualizing on the fibres via the metric on the second factor of $\not{S}_X^+ = X\times\mathbb{H}$, the coefficients of this 2-form can be represented by a real $\operatorname{ad}\tilde{P}$-valued 4×4-matrix

$$\Xi := \begin{pmatrix} \nabla_1\phi'_5 & \nabla_2\phi'_5 & \nabla_3\phi'_5 & \nabla_4\phi'_5 \\ \nabla_1\phi'_6 & \nabla_2\phi'_6 & \nabla_3\phi'_6 & \nabla_4\phi'_6 \\ \nabla_1\phi'_7 & \nabla_2\phi'_7 & \nabla_3\phi'_7 & \nabla_4\phi'_7 \\ \nabla_1\phi'_8 & \nabla_2\phi'_8 & \nabla_3\phi'_8 & \nabla_4\phi'_8 \end{pmatrix}$$

with respect to basis $\mathrm{d}x^1,\dots,\mathrm{d}x^4$ of $\mathrm{pr}_1^*\,\mathrm{T}^*X$ in the rows and the basis $\mathrm{d}x^5,\dots,\mathrm{d}x^8$ of $\mathrm{pr}_2^*\,\mathrm{T}\mathbb{H}$ in the columns. This matrix represents an element of $\mathrm{T}^*X\otimes_{\mathbb{R}}\mathrm{T}\mathbb{H}\cong\mathrm{Hom}_{\mathbb{R}}(\mathrm{T}X,\mathrm{T}\mathbb{H})$ and the projection π''_4 corresponds to the projection onto $\mathrm{Hom}_{\mathbb{H}}(\mathrm{T}X;\mathrm{T}\mathbb{H})$.

Being a Hyperkähler manifold, the K3 surface X has a quaternionic structure (J_1,J_2,J_3) which allows to identify $\mathrm{T}_pX\cong\mathbb{H}$ for any $p\in X$. Likewise, T^*X is a (non-trivial) $\mathbb{H}^*$-bundle over the K3 surface X. If the quaternionic structure on $\mathrm{T}X$ is induced by left $\mathbb{H}$-multiplication, the same is also true for the dual tangent bundle T^*X. Therefore, let the coordinates on X be chosen such that $\mathrm{d}x^5,\dots,\mathrm{d}x^8$ are identified with the dual of the canonical $\mathbb{H}$-basis $1,\mathrm{i},\mathrm{j},\mathrm{k}$. Using the matrices $\ell_\mathrm{i},\ell_\mathrm{j},\ell_\mathrm{k}$ of left quaternionic multiplication specified in app. A.6 it follows

$$\begin{aligned}\pi_{\mathbb{H}}(\Xi) &= \frac{1}{4}\big(\Xi-\mathcal{C}(\Xi)\big) = \frac{1}{4}\left(\Xi-\ell_\mathrm{i}\Xi\ell_\mathrm{i}-\ell_\mathrm{j}\Xi\ell_\mathrm{j}-\ell_\mathrm{k}\Xi\ell_\mathrm{k}\right)\\ &= \frac{1}{4}\begin{pmatrix} \nabla_1\phi'_5+\nabla_2\phi'_6+\nabla_3\phi'_7+\nabla_4\phi'_8 & \nabla_2\phi'_5-\nabla_1\phi'_6-\nabla_4\phi'_7+\nabla_3\phi'_8 & \nabla_3\phi'_5+\nabla_4\phi'_6-\nabla_1\phi'_7-\nabla_2\phi'_8 & \nabla_4\phi'_5-\nabla_3\phi'_6+\nabla_2\phi'_7-\nabla_1\phi'_8 \\ \nabla_1\phi'_6-\nabla_2\phi'_5-\nabla_3\phi'_8+\nabla_4\phi'_7 & \nabla_2\phi'_6+\nabla_1\phi'_5+\nabla_4\phi'_8+\nabla_3\phi'_7 & \nabla_3\phi'_6-\nabla_4\phi'_5+\nabla_1\phi'_8-\nabla_2\phi'_7 & \nabla_4\phi'_6+\nabla_3\phi'_5-\nabla_2\phi'_8-\nabla_1\phi'_7 \\ \nabla_1\phi'_7+\nabla_2\phi'_8-\nabla_3\phi'_5-\nabla_4\phi'_6 & \nabla_2\phi'_7-\nabla_1\phi'_8+\nabla_4\phi'_5-\nabla_3\phi'_6 & \nabla_3\phi'_7+\nabla_4\phi'_8+\nabla_1\phi'_5+\nabla_2\phi'_6 & \nabla_4\phi'_7-\nabla_3\phi'_8-\nabla_2\phi'_5+\nabla_1\phi'_6 \\ \nabla_1\phi'_8-\nabla_2\phi'_7+\nabla_3\phi'_6-\nabla_4\phi'_5 & \nabla_2\phi'_8+\nabla_1\phi'_7-\nabla_4\phi'_6-\nabla_3\phi'_5 & \nabla_3\phi'_8-\nabla_4\phi'_7-\nabla_1\phi'_6+\nabla_2\phi'_5 & \nabla_4\phi'_8+\nabla_3\phi'_7+\nabla_2\phi'_6+\nabla_1\phi'_5 \end{pmatrix}\\ &= \frac{1}{4}\left(\mathbb{1}\,\nabla_1-\ell_\mathrm{i}\nabla_2-\ell_\mathrm{j}\nabla_3-\ell_\mathrm{k}\nabla_4\right)\begin{pmatrix}\phi'_5\\ \phi'_6\\ \phi'_7\\ \phi'_8\end{pmatrix}.\end{aligned}$$

If the basis $\{1,\mathrm{i},\mathrm{j},\mathrm{k}\}$ in (4.23) is replaced with the quaternionic matrices $\{\mathbb{1},\ell_\mathrm{i},\ell_\mathrm{j},\ell_\mathrm{k}\}$, then the previous result is obtained up to the constant factor $\frac{1}{4}$. More precisely, this identification corresponds to the isomorphism

$$\begin{aligned}\kappa_3 : \mathrm{Hom}_{\mathbb{H}}(R,S) &\xrightarrow{\cong}\mathbb{H}\\ \mathbb{1}\,a+\ell_\mathrm{i}b+\ell_\mathrm{j}c+\ell_\mathrm{k}d &\mapsto a+\mathrm{i}b+\mathrm{j}c+\mathrm{k}d.\end{aligned}$$

Using the bundle extensions of the isomorphisms κ_1, κ_2 and the above κ_3 from the explicit construction of the projection π''_4 in sec. 4.4.2, it follows

$$\kappa_3\circ\pi_{\mathbb{H}}\circ\kappa_2\circ\kappa_1(\mathrm{d}_{A'}\Phi') = \frac{1}{4}\bar{\nabla}^{\mathbb{H}}\Phi' = \frac{1}{4}\,\mathrm{pr}_1^*\,\bar{\nabla}^{\mathbb{H}}\Phi$$

by the previous computations. Therefore, the second equation is simplified according to

$$\begin{aligned}\pi''_4(\mathrm{d}_{A'}\Phi') = \tilde{\zeta}\circ\kappa_3\circ\pi_{\mathbb{H}}\circ\kappa_2\circ\kappa_1(\mathrm{d}_{A'}\Phi') = \tilde{\zeta}\left(\frac{1}{4}\,\mathrm{pr}_1^*\,\bar{\nabla}^{\mathbb{H}}\Phi\right) = 0\\ \iff \bar{\nabla}^{\mathbb{H}}\Phi = 0.\end{aligned}$$

By the explicit representation of the four-dimensional Clifford algebra (cf. app. A.4 for reference) it follows that $\bar{\nabla}^{\mathbb{H}}$ can be identified with the Dirac operator on positive spinors and therefore the second equation is equivalent to

$$\not{D}_A^+\Phi = 0,$$

where A refers to the induced connection on $\operatorname{ad} P$ since $\Phi \in \Gamma(\not{S}_X^+ \otimes \operatorname{ad} P)$ is a positive spinor with values in the adjoint bundle. □

Solutions to $\not{D}\Phi = 0 \iff \Phi \in \ker \not{D}$ are called **harmonic spinors**, which are related to the scalar (Ricci-)curvature of the base space, cf. [LM89, p.160]. Therefore, the theorem proves that all Spin(7)-instantons on the positive spinor bundle's total space are described by a harmonic adjoint (positive) spinor $\Phi \in \Gamma(\not{S}_X^+ \otimes \operatorname{ad} P)$ and a four-dimensional gauge field determined by $F_A^+ = \left[\Phi, \bar{\Phi}\right]$ up to a constant factor.

CHAPTER 5

Outlook

In the previous chapter two examples of dimensional reduction were investigated in detail. The reduction of a Spin(7)-instanton in temporal gauge on the product space $\mathbb{R} \times Z$ with Z being a G_2-manifold, which produced the inhomogeneous G_2-instanton

$$\pi_7^{G_2}(F_{A_t}) = \eta(F_{A_t}).$$

This can be regarded as an expected result considering the four-dimensional case, where the Bogomolny monopole equations were obtained upon reduction to three dimensions. However, in three dimensions the result of ex. 2.34, i.e. the lower-dimensional formulation of a four-dimensional instanton, provides the basis of Floer theory. Furthermore, in four dimensions the instanton equation on $\mathbb{R} \times M^3$ can be interpreted as a sort of gradient flow of the Chern-Simons functional, which is defined on the space of gauge-equivalence classes of connections. According to [DT98, §3] a similar interpretation is also viable for the result derived in sec. 4.1. See [Don02] for an introduction to Floer theory.

The second dimensional reduction was carried out on the product space $X \times \mathbb{R}^4$, where X is a K3 surface of real dimension four. Since any K3 surface is both Hyperkähler and Calabi-Yau, i.e. Ricci-flat and equipped with a quaternionic structure on its tangent bundle, the space $X \times \mathbb{R}^4$ is identified with the total space of the (trivial) positive spinor bundle $\not{S}_X^+ = X \times \mathbb{H}$. As it was shown in thm. 4.25, the translation-invariant instantons in this situation are described by the four-dimensional equations

$$\begin{cases} F_A^+ = \left[\Phi, \bar{\Phi}\right] \\ \not{D}_A^+ \Phi = 0 \end{cases}$$

The result of this dimensional reduction shares a striking resemblance to the well-known four-dimensional Seiberg-Witten equations

$$\begin{cases} F_A^+ = \sigma(\psi) \\ \not{D}^+ \psi = 0, \end{cases}$$

where $\sigma : \not{S}^+ \longrightarrow \Lambda_+^2$ is a sort of traceless square of the positive spinor $\psi \in \Gamma(\not{S}^+)$. More research is necessary to identify deeper links between the two sets of equations and the corresponding gauge theories, the only difference being found in the relation of the gauge field (i.e. connection) to the spinor.

There are many other setups which may be studied in a similar fashion. For example, if Y is a Calabi-Yau 3-fold (i.e. a real manifold of dimension six with holonomy SU(3)), the translation-invariant Spin(7)-instantons on $Y \times \mathbb{R}^2$ can be considered using complex methods. In the dimensional reduction to six dimensions one obtains the Donaldson-Uhlenbeck-Yau equations, which are relevant to the description of the moduli space of stable holomorphic vector bundles. See [BKS98, §4.4.1] for a computation of this result and an account on the physical relevance of such dimensional reductions in the context of supersymmetric Yang-Mills theory.

APPENDIX A

Calculations, definitions and reference

A.1. The operator $\star_\Omega$ in local coordinates

Let $\Omega_0^{\mathbb{H}} \in \Lambda^2(\mathbb{R}^8)^*$ be the 4-form in the quaternionic formulation, cf. def. 4.11, which is used in the definition in Spin(7) and the eigenspace splitting $\Lambda^2(\mathbb{R}^8)^* = \Lambda^2_7 \oplus \Lambda^2_{21}$. Define the linear operator $\star_\Omega := \star(\Omega_0^{\mathbb{H}} \wedge .)$ that restricts to an endomorphism of $\Lambda^2(\mathbb{R}^8)^*$. With respect to the canonical basis $\{\mathrm{d}x^{ij} : 1 \le i < j \le 8\}$ and the standard Euclidean inner product on $\mathbb{R}^8$, the operator can be expressed in terms of this basis. The grouping of the basis corresponds to the seven eigenvectors from lem. 4.13:

$$1\begin{cases} \star_\Omega(\mathrm{d}x^{12}) = \mathrm{d}x^{34} - \mathrm{d}x^{56} - \mathrm{d}x^{78} \\ \star_\Omega(\mathrm{d}x^{34}) = \mathrm{d}x^{12} - \mathrm{d}x^{78} - \mathrm{d}x^{56} \\ \star_\Omega(\mathrm{d}x^{56}) = \mathrm{d}x^{78} - \mathrm{d}x^{34} - \mathrm{d}x^{12} \\ \star_\Omega(\mathrm{d}x^{78}) = \mathrm{d}x^{56} - \mathrm{d}x^{34} - \mathrm{d}x^{12} \end{cases}$$

$$2\begin{cases} \star_\Omega(\mathrm{d}x^{13}) = -\mathrm{d}x^{24} + \mathrm{d}x^{68} - \mathrm{d}x^{57} \\ \star_\Omega(\mathrm{d}x^{24}) = -\mathrm{d}x^{13} - \mathrm{d}x^{68} + \mathrm{d}x^{57} \\ \star_\Omega(\mathrm{d}x^{57}) = -\mathrm{d}x^{68} + \mathrm{d}x^{24} - \mathrm{d}x^{13} \\ \star_\Omega(\mathrm{d}x^{68}) = -\mathrm{d}x^{57} - \mathrm{d}x^{24} + \mathrm{d}x^{13} \end{cases}$$

$$3\begin{cases} \star_\Omega(\mathrm{d}x^{14}) = \mathrm{d}x^{23} - \mathrm{d}x^{67} - \mathrm{d}x^{58} \\ \star_\Omega(\mathrm{d}x^{23}) = \mathrm{d}x^{14} - \mathrm{d}x^{67} - \mathrm{d}x^{58} \\ \star_\Omega(\mathrm{d}x^{58}) = \mathrm{d}x^{67} - \mathrm{d}x^{23} - \mathrm{d}x^{14} \\ \star_\Omega(\mathrm{d}x^{67}) = \mathrm{d}x^{58} - \mathrm{d}x^{23} - \mathrm{d}x^{14} \end{cases}$$

$$4\begin{cases} \star_\Omega(\mathrm{d}x^{15}) = \mathrm{d}x^{26} + \mathrm{d}x^{37} + \mathrm{d}x^{48} \\ \star_\Omega(\mathrm{d}x^{26}) = \mathrm{d}x^{15} + \mathrm{d}x^{48} + \mathrm{d}x^{37} \\ \star_\Omega(\mathrm{d}x^{37}) = \mathrm{d}x^{48} + \mathrm{d}x^{15} + \mathrm{d}x^{26} \\ \star_\Omega(\mathrm{d}x^{48}) = \mathrm{d}x^{37} + \mathrm{d}x^{26} + \mathrm{d}x^{15} \end{cases}$$

$$5\begin{cases} \star_\Omega(\mathrm{d}x^{25}) = -\mathrm{d}x^{16} - \mathrm{d}x^{47} + \mathrm{d}x^{38} \\ \star_\Omega(\mathrm{d}x^{16}) = -\mathrm{d}x^{25} - \mathrm{d}x^{38} + \mathrm{d}x^{47} \\ \star_\Omega(\mathrm{d}x^{47}) = -\mathrm{d}x^{38} - \mathrm{d}x^{25} + \mathrm{d}x^{16} \\ \star_\Omega(\mathrm{d}x^{38}) = -\mathrm{d}x^{47} - \mathrm{d}x^{16} + \mathrm{d}x^{25} \end{cases}$$

$$6\begin{cases} \star_\Omega(\mathrm{d}x^{35}) = \mathrm{d}x^{46} - \mathrm{d}x^{17} - \mathrm{d}x^{28} \\ \star_\Omega(\mathrm{d}x^{46}) = \mathrm{d}x^{35} - \mathrm{d}x^{28} - \mathrm{d}x^{17} \\ \star_\Omega(\mathrm{d}x^{17}) = \mathrm{d}x^{28} - \mathrm{d}x^{35} - \mathrm{d}x^{46} \\ \star_\Omega(\mathrm{d}x^{28}) = \mathrm{d}x^{17} - \mathrm{d}x^{46} - \mathrm{d}x^{35} \end{cases}$$

$$7\begin{cases} \star_\Omega(\mathrm{d}x^{45}) = -\mathrm{d}x^{36} + \mathrm{d}x^{27} - \mathrm{d}x^{18} \\ \star_\Omega(\mathrm{d}x^{36}) = -\mathrm{d}x^{45} + \mathrm{d}x^{18} - \mathrm{d}x^{27} \\ \star_\Omega(\mathrm{d}x^{27}) = -\mathrm{d}x^{18} + \mathrm{d}x^{45} - \mathrm{d}x^{36} \\ \star_\Omega(\mathrm{d}x^{18}) = -\mathrm{d}x^{27} + \mathrm{d}x^{36} - \mathrm{d}x^{45} \end{cases}$$

Those results for the basis elements are rearranged into the 28×28 operator matrix $\Xi^{\mathbb{H}}$ of $\star_\Omega = \star(\Omega_0^{\mathbb{H}} \wedge .)$, which is shown in tab. A.1. Via the established methods for computing the determinant of large matrices containing mostly zeros, the characteristic polynomial

$$\chi(\star_\Omega; \lambda) = \det(\Xi - \lambda\, \mathbb{1}_{28}) = (\lambda+1)^{21}(\lambda-3)^7$$

is computed as mentioned in lem. 3.29. Note that the double permutation of basis elements $x^1 \leftrightarrow x^2$ and $x^3 \leftrightarrow x^4$ that relates Ω_0 to $\Omega_0^{\mathbb{H}}$ has determinant 1.

	12	**13**	**14**	**15**	**16**	**17**	**18**	**23**	**24**	**25**	**26**	**27**	**28**	**34**	**35**	**36**	**37**	**38**	**45**	**46**	**47**	**48**	**56**	**57**	**58**	**67**	**68**	**78**
12		0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	-1	0	0	0	0	-1
13	0		0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	1	0
14	0	0		0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	0	0
15	0	0	0		0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0
16	0	0	0	0		0	0	0	0	-1	0	0	0	0	0	0	0	-1	0	0	1	0	0	0	0	0	0	0
17	0	0	0	0	0		0	0	0	0	0	0	1	0	-1	0	0	0	0	-1	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0		0	0	0	0	-1	0	0	0	1	0	0	-1	0	0	0	0	0	0	0	0	0
23	0	0	1	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	0	0
24	0	-1	0	0	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	-1	0
25	0	0	0	0	-1	0	0	0	0		0	0	0	0	0	0	0	1	0	0	-1	0	0	0	0	0	0	0
26	0	0	0	1	0	0	0	0	0	0		0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0
27	0	0	0	0	0	0	-1	0	0	0	0		0	0	0	-1	0	0	1	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	1	0	0	0	0	0	0		0	-1	0	0	0	0	-1	0	0	0	0	0	0	0	0
34	1	0	0	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0	0	-1	0	0	0	0	-1
35	0	0	0	0	0	-1	0	0	0	0	0	0	-1	0		0	0	0	0	1	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	1	0	0	0	0	-1	0	0	0		0	0	-1	0	0	0	0	0	0	0	0	0
37	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0		0	0	0	0	1	0	0	0	0	0	0
38	0	0	0	0	-1	0	0	0	0	1	0	0	0	0	0	0	0		0	0	-1	0	0	0	0	0	0	0
45	0	0	0	0	0	0	-1	0	0	0	0	1	0	0	0	-1	0	0		0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	-1	0	0	0	0	0	0	-1	0	1	0	0	0	0		0	0	0	0	0	0	0	0
47	0	0	0	0	1	0	0	0	0	-1	0	0	0	0	0	0	0	-1	0	0		0	0	0	0	0	0	0
48	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0		0	0	0	0	0	0
56	-1	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0		0	0	0	0	1
57	0	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0		0	0	-1	0
58	0	0	-1	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		1	0	0
67	0	0	-1	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1		0	0
68	0	1	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0		0
78	-1	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	1	0	0	0	0	

TABLE A.1. The symmetric operator matrix to $\star_\Omega : \Lambda^2(\mathbb{R}^8)^* \longrightarrow \Lambda^2(\mathbb{R}^8)^*$ with respect to the basis $\{\mathrm{d}x^{ij} : 1 \leq i < j \leq 8\}$ with diagonal zeros omitted.

A.2. Fibre and vector bundles

The geometrical framework of gauge theories is provided by fibre bundles. The general theory of fibre bundles is well covered in the literature, e.g. [Hus98].

DEFINITION A.1. A **fibre bundle** is a collection (E, M, π, F) of objects satisfying the following properties:

- The **total space** E, the **base space** M and the **fibre** F are topological manifolds.
- The **bundle projection** $\pi : E \twoheadrightarrow M$ is a continuous surjective mapping, such that the condition of **local triviality** is fulfilled: For any point $x \in M$ there exists an open neighbourhood $U \subseteq M$ and a homeomorphism $\phi_U : \pi^{-1}(U) \xrightarrow{\approx} U \times F$ that commutes with the bundle projection, i.e.

$$\begin{array}{ccc} E|_U := \pi^{-1}(U) & \xrightarrow[\approx]{\phi_U} & U \times F \\ & {\scriptstyle\pi}\searrow \quad \circlearrowleft \quad \swarrow{\scriptstyle \mathrm{pr}_1 : (a,b) \mapsto a} & \\ & U & \end{array} \qquad \rightsquigarrow \qquad \mathrm{pr}_1 \circ \phi_U = \pi|_{E|_U}.$$

  This homeomorphism ϕ_U is called the **local trivialization** and essentially provides a local product structure on the bundle's total space.

A fibre bundle is often denoted as $E \xrightarrow{\pi} M$ for short, with all conditions of the above definition implied. By taking the fibre F to be a space with additional properties, certain special types of fibre bundles are obtained. Consider a group G such that the basic operations

$$\begin{aligned} \text{mult} : G \times G &\longrightarrow G \\ (g, g') &\mapsto gg' \end{aligned} \qquad \text{and} \qquad \begin{aligned} \text{inv} : G &\longrightarrow G \\ g &\mapsto g^{-1} \end{aligned}$$

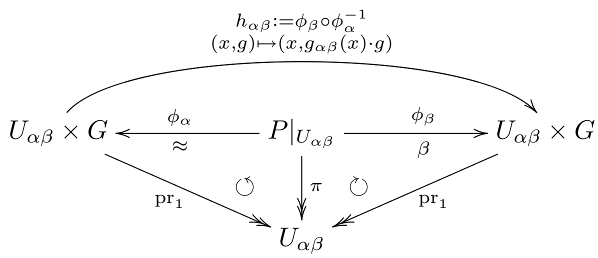
are continuous mappings, which is called a **topological group**.

Definition A.2. A **principal G-bundle** is a fibre bundle (P, M, π, G) with a continuous right action $P \times G \longrightarrow P$ satisfying the following properties:

- The action preserves the fibres, i.e. $r_g(P_x) = P_x$, where $r_g : h \mapsto hg$ denotes right multiplication on the fibre $G \cong P_x$ for any point $x \in M$ of the base space.
- The action is **free**, i.e. if $p \cdot g = p$ holds for all $g \in G$ and $p \in P$ then it follows that the group element is $g = e \in G$.
- The action is **transitive**, i.e. for any two $p, q \in P$ there exists a $g \in G$ such that $p \cdot g = q$.

Equivalently, the action is said to be **regular**, i.e. for any two point $p, q \in P$ of the total space there exists an unique group element $g \in G$ such that $p \cdot g = q$ holds. This turns $P \xrightarrow{\pi} M$ into a right G-module.

The bundle structure is equivalently described by a collection of **transition functions** $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G$ which relate the different local patches of the bundle on domain overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta \subset M$ as shown in the following diagram:



Another possibility is to attach a vector space at each point of the base space and to equip the bundle with a certain kind of linear structure.

Definition A.3. Let $V = \mathbb{R}^n$ or $\mathbb{C}^n$ be a real or complex vector space of dimension n. A **vector bundle** of rank n is a fibre bundle (W, M, π, V), such that any local trivialization ϕ_U restricts to a (real or complex) linear isomorphism $\phi_U|_x : W_x \xrightarrow{\cong} \{x\} \times V \cong V$ for any fibre $W_x := \pi^{-1}(x)$ of the vector bundle.

Given a representation of the group of a principal G-bundle on another space, one constructs a new fibre bundle that is invariant under the representation of the G-action.

Definition A.4. Let $P \xrightarrow{\pi} M$ be a principal G-bundle and F a topological space with a representation $\rho : G \longrightarrow \mathrm{Aut}(F)$ of the bundle group G. On the product space $P \times F$ there exists a free right G-action $(p, f) \cdot g := \big(p \cdot g, \rho_{g^{-1}}(f)\big)$. The right quotient space $P \times_G F := (P \times F)/G$ is called the **associated fibre bundle** of P and F, where the new bundle projection $\tilde{\pi} : P \times_G F \twoheadrightarrow M$ is given by $\tilde{\pi}([p, f]) := \pi(p)$.

Notation. Instead of $P \times_G F$ one also encounters the notation $P \times_\rho F$ which specifies the used representation $\rho : G \longrightarrow \mathrm{Aut}(F)$ in order to avoid any ambiguities.

In order to show the bundle structure of $P \times_G F$, one needs to show the existence of local trivializations and specify an appropiate topology, see [KN63, p. 54] for details. Usually the new fibre F is a vector space and the action of G onto F is provided by a linear representation, such that $P \times_G F$ is called the associated vector bundle in those cases.

A.3. Clifford algebras and spinors

In the second dimensional reduction the Spin(7)-base space corresponds to the total space of a (trivial) spinor bundle. Therefore, a short summary of spinors on Hyperkähler surfaces is provided for reference. For the general theory of spinors see the book [LM89] of Lawson and Michelsohn.

Definition A.5. Let $V = \mathbb{R}^n$ be the n-dimensional real Euclidean vector space and $e_1, \ldots, e_n \in V$ be an orthonormal basis, which are subject to the algebraic relations

$$\{e_i, e_j\} = e_i e_j + e_j e_i = -2\delta_{ij}. \tag{A.1}$$

The real 2^n-dimensional algebra generated by those elements is called the (real) **Clifford algebra** $\mathrm{C}\ell(n)$.[a]

The original vector space V is a canonical subspace $V \subset \mathrm{C}\ell(n)$. Each Clifford algebra can be understood as a (possibly $\mathbb{Z}_2$-graded) matrix algebra over either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Definition A.6. Let $\alpha : v \mapsto -v$ be the mirror mapping in V and let $\tilde{\alpha} : \mathrm{C}\ell(n) \longrightarrow \mathrm{C}\ell(n)$ denote the corresponding extension to the Clifford algebra. Due to $\tilde{\alpha}^2 = \mathrm{Id}$, there exists a corresponding eigenspace splitting

$$\mathrm{C}\ell(n) = \mathrm{C}\ell^0(n) \oplus \mathrm{C}\ell^1(n),$$

where $\mathrm{C}\ell^0(n) := \mathrm{Eig}(\tilde{\alpha}; +1)$ is called the **even part** and $\mathrm{C}\ell^1(n) := \mathrm{Eig}(\tilde{\alpha}; -1)$ the **odd part**.

Both the even and odd part of $\mathrm{C}\ell(n)$ are subalgebras, that are generated by products of either an even or odd number of basis elements e_i. Let $\omega := e_1 e_2 \cdots e_n \in \mathrm{C}\ell(n)$ denote the **volume element** with respect to the Clifford multiplication. If $\omega^2 = 1$ holds,

$$\mathrm{C}\ell(n) = \mathrm{C}\ell^+(n) \oplus \mathrm{C}\ell^-(n)$$

provides a second kind of eigenspace decomposition, where $\mathrm{C}\ell^{\pm}(n) := \mathrm{Eig}(\omega; \pm 1)$ are called the **positive** and **negative subalgebra**, respectively.

Given a Clifford algebra $\mathrm{C}\ell(n)$, there is a natural subgroup of invertable elements defined by

$$\mathrm{C}\ell^{\times}(n) := \big\{\varphi \in \mathrm{C}\ell(n) : \text{there exists an inverse } \varphi^{-1} \text{ with } \varphi^{-1}\varphi = \varphi\varphi^{-1} = 1\big\}.$$

In fact, $\mathrm{C}\ell^{\times}(n)$ is a Lie group of dimension 2^n, where the Lie algebra corresponds to the original Clifford algebra, i.e. $\mathrm{c}\ell^{\times}(n) = \mathrm{C}\ell(n)$.

Definition A.7. Let $V = \mathbb{R}^n$ be the n-dimensional real vector space. Then

$$\mathrm{Spin}(n) := \big\{\text{products } v_1 \cdots v_r \in \mathrm{C}\ell^{\times}(n) : \|v_i\| = 1 \text{ for all } v_i \in V\big\}$$

is called the **spin group** associated to the (real) Clifford algebra $\mathrm{C}\ell(n)$.

Lemma A.8. *Let* $\mathrm{SO}(n)$ *denote the special orthogonal group on* $\mathbb{R}^n$. *Then*

$$0 \dashrightarrow \mathbb{Z}_2 \hookrightarrow \mathrm{Spin}(n) \twoheadrightarrow \mathrm{SO}(n) \dashrightarrow 0$$

is a short exact sequence, which implies $\mathrm{SO}(n) \cong \mathrm{Spin}(n)/\mathbb{Z}_2$, *i.e. the spin group is the two-fold covering group to* $\mathrm{SO}(n)$.

Proof. See [LM89, thm. I.2.10] for the proof and further details. □

Furthermore, for $n > 2$ each spin group $\mathrm{Spin}(n)$ is the universal cover to $\mathrm{SO}(n)$. The previous lemma also implies $\mathfrak{spin}(n) \cong \mathfrak{so}(n)$ for the Lie algebras.

Since each Clifford algebra corresponds to a matrix algebra of type $A = \mathrm{Mat}_{m\times m}(F)$ or $B = \mathrm{Mat}_{m\times m}(F) \oplus \mathrm{Mat}_{m\times m}(F)$ for $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, there are canonical irreducible representations on $S := F^m$. In the case A this is simply the standard representation of the matrix. For the case B there are two inequivalent representations induced by the standard representation of either the first or second summand.

[a]See [LM89] for the general definition on non-Euclidean inner product spaces.

DEFINITION A.9. Let $\rho_n : \mathrm{C}\ell(n) \longrightarrow \mathrm{End}_F(S)$ be an irreducible (standard) representation of the Clifford algebra and $\mathrm{Spin}(n) \subset \mathrm{C}\ell^0(n) \subset \mathrm{C}\ell(n)$ be the corresponding spin group. Define

$$\Delta_n := \rho_n\big|_{\mathrm{Spin}(n)} : \mathrm{Spin}(n) \longrightarrow \mathrm{End}_F(S)$$

by restriction, which is called the **spinor representation** of $\mathrm{Spin}(n)$.

The classification of spinor representations can be derived from the classification of Clifford algebras $\mathrm{C}\ell(n-1)$ due to $\mathrm{C}\ell^0(n) \cong \mathrm{C}\ell(n-1)$, see [LM89, thm. 3.7] and [LM89, §I.4]. In particular, the decomposition $\mathrm{C}\ell(n) = \mathrm{C}\ell^+(n) \oplus \mathrm{C}\ell^-(n)$ into the positive and negative eigenspace of the volume form extends to the spinor representation, such that $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ follows if $\omega^2 = 1$. This construction is now lifted from algebras and vector spaces to bundles.

DEFINITION A.10. Let (M, g) be an oriented Riemannian manifold of dimension n and $Q \subset \mathrm{Fr}(M)$ the canonical $\mathrm{SO}(n)$-structure of the metric and orientation. A principal $\mathrm{Spin}(n)$-bundle $\tilde{Q} \xrightarrow{\tilde{\pi}} M$ is called a **spin structure**, if there is a two-sheeted covering map $\tilde{Q} \longrightarrow Q$.

The existence of spin structures is restricted by the topology of the base space M and is equivalent to the vanishing of the second Stiefel-Whitney class $w_2(M)$, see [LM89, §II.1]. A manifold admitting a spin structure on its tangent bundle is called a **spin manifold**.

DEFINITION A.11. Let $\tilde{Q} \xrightarrow{\tilde{\pi}} M$ be a spin structure on the n-dimensional manifold M and $\Delta_n : \mathrm{Spin}(n) \longrightarrow \mathrm{End}_F(S)$ a spinor representation of $\mathrm{Spin}(n)$. The associated bundle

$$\$_M := \tilde{Q} \times_{\Delta_n} S$$

is called the (real) **spinor bundle** and its sections $\psi \in \Gamma(\$_M)$ are called **spinors**.

Again, if $\omega^2 = 1$ is satisfied there is the decomposition $\$_M = \$_M^+ \oplus \$_M^-$ into eigenspaces of the volume element, and the sections $\psi^\pm \in \Gamma(\$_M^\pm)$ are called **positive** or **negative spinors**, respectively.

Since the two-sheeted covering map $\tilde{Q} \longrightarrow Q$ of the spin structure is a local diffeomorphism, the means of differentiation on $\$_M$ are derived from the $\mathrm{SO}(n)$-structure Q, i.e. from the tangent bundle's Levi-Civita connection in case of a Riemannian manifold. With respect to this **spin connection**, a spinor $\psi \in \Gamma(\$_M)$ is called **parallel** if $\psi \neq 0$ and $\nabla\psi = 0$ holds everywhere. The number of linearly independent parallel spinors is related to the triviality of the spinor bundle. Due to Wang's theorem (see [Wan89]), this result is known for all Riemannian manifolds appearing in Berger's classification, cf. [Joy00, thm. 3.6.1] for a more concise summary of the result.

There is a natural Clifford multiplication defined in each tangent space $\mathrm{T}_x M$, as the spinor representation Δ_n used in the definition of any spinor bundle actually comes from a restriction of a representation of the Clifford algebra.

DEFINITION A.12. Let $\$_M \xrightarrow{\pi} M$ be the spinor bundle on the n-dimensional Riemannian manifold (M, g) and ∇ the spin connection. For $x \in M$ let $e_1, \dots, e_n \in \mathrm{T}_x M$ be an orthonormal basis. Define a mapping $\not{D} : \Gamma(\$_M) \longrightarrow \Gamma(\$_M)$ pointwise by

$$(\not{D}\sigma)_x := \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \sigma(x),$$

where "$\cdot$" denotes the Clifford multiplication in each tangent space. This first-order linear differential operator is called the **Dirac operator** of the spinor bundle.

A.4. The four-dimensional Clifford algebra

Consider the spinor representations in four dimensions. From the relations (A.1) of Clifford algebras it follows $\mathrm{C}\ell(4) \cong \mathrm{Mat}_{2\times 2}(\mathbb{H})$, which has a canonical standard representation on $S = \mathbb{H}^2$. The even subalgebra that contains $\mathrm{Spin}(4)$ is $\mathrm{C}\ell^0(4) \cong \mathrm{C}\ell(3) \cong \mathbb{H} \oplus \mathbb{H}$, which corresponds to the diagonal matrices of $\mathrm{Mat}_{2\times 2}(\mathbb{H})$.

The even part of $C\ell(4)$ is isomorphic to the vector space $\Lambda^0\mathbb{R}^4 \oplus \Lambda^2\mathbb{R}^4 \oplus \Lambda^4\mathbb{R}^4$, i.e. $C\ell^0(4)$ is spanned by elements

$$\begin{array}{ccccc} & f_1 := e_1e_2, & f_2 := e_1e_3, & f_3 := e_1e_4, & \\ 1, & & e_2e_3, & e_2e_4, & e_1e_2e_3e_4 \\ & & & e_3e_4, & \end{array}$$

and generated by the three elements f_1, f_2, f_3 defined above, such that $C\ell^0(4) \cong C\ell(3)$ is given explicitly as the three generators satisfying the relations (A.1). Motivated by the splitting of 2-forms $\Lambda^2 = \Lambda^2_+\mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4$, consider the usual (anti-)self-dual basis in terms of the generators of the even part

$$\begin{aligned} e_1e_2 \pm e_3e_4 &= f_1 \pm f_2f_3 \\ e_1e_3 \pm e_4e_2 &= f_2 \pm f_3f_1 \\ e_1e_4 \pm e_2e_3 &= f_3 \pm f_1f_2. \end{aligned}$$

Fixing some signs, this induces the splitting $C\ell(3) \cong \mathbb{H}_+ \oplus \mathbb{H}_-$ explicitly: The isomorphism between the two copies $\mathbb{H}_+$ and $\mathbb{H}_-$ of the quaternions is given by

$$\begin{aligned} \mathbb{H}_+ : \{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\} &\mapsto \frac{1}{2}\Big\{1 + f_1f_2f_3, f_2f_3 - f_1, f_3f_1 - f_2, f_1f_2 - f_3\Big\}, \\ \mathbb{H}_- : \{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\} &\mapsto \frac{1}{2}\Big\{1 - f_1f_2f_3, f_2f_3 + f_1, f_3f_1 + f_2, f_1f_2 + f_3\Big\} \end{aligned} \tag{A.2}$$

which can be seen from checking the relations (4.5). Furthermore, one easily checks that the volume element $\omega = f_1f_2f_3$ applied to the above basis elements of $\mathbb{H}_+$ corresponds to Id, whereas it gives $-\mathrm{Id}$ on $\mathbb{H}_-$. Therefore the decomposition $C\ell(3) = C\ell^+(3) \oplus C\ell^-(3)$ is represented and subsequently $\Delta_4 = \Delta_4^+ \oplus \Delta_4^-$.

The above constructions can be made explicit in terms of matrices as follows: By checking the Clifford algebra relations (A.1), the generators e_1, e_2, e_3, e_4 of $C\ell(4)$ are represented by the quaternionic 2×2-matrices[b]

$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 & \mathrm{j} \\ \mathrm{j} & 0 \end{pmatrix} \qquad e_4 = \begin{pmatrix} 0 & \mathrm{k} \\ \mathrm{k} & 0 \end{pmatrix} \tag{A.3}$$

and the generators f_1, f_2, f_3 of the even subalgebra $C\ell(3) \cong C\ell^0(4) \subset C\ell(4)$ are the matrices

$$f_1 = e_1e_2 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \qquad f_2 = e_1e_3 = \begin{pmatrix} -\mathrm{j} & 0 \\ 0 & \mathrm{j} \end{pmatrix} \qquad f_3 = e_1e_4 = \begin{pmatrix} -\mathrm{k} & 0 \\ 0 & \mathrm{k} \end{pmatrix}.$$

The volume element ω_4 and the corresponding projection operators onto the positive and negative subspace are given by

$$\omega = f_1f_2f_3 = \begin{pmatrix} -\mathrm{ijk} & 0 \\ 0 & \mathrm{ijk} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} P_+ &= \frac{1}{2}(\mathbb{1} + \omega_4) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ P_- &= \frac{1}{2}(\mathbb{1} - \omega_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Those projection matrices are equal to the unit 1 in $\mathbb{H}_+$ and $\mathbb{H}_-$, respectively. More precisely, with respect to the chosen matrix representation of the Clifford algebra, the isomorphism (A.2) is given by

$$\begin{aligned} \mathbb{H}_+ : \{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\} &\mapsto \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{i} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{j} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{k} & 0 \\ 0 & 0 \end{pmatrix} \right\} \\ \mathbb{H}_- : \{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\} &\mapsto \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{j} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{k} \end{pmatrix} \right\}, \end{aligned}$$

[b] In physics those matrices are called the Euclidean Dirac γ-matrices, which are preferable to the Lorentz-signature Dirac matrices for certain computations of Feynman path integrals.

i.e. the even subalgebra is in fact given by the diagonal matrices of $\mathrm{Mat}_{2\times 2}(\mathbb{H})$. Therefore, the spin group in four dimensions is of a particular simple structure, see the next section.

Furthermore, let ∇_i denote the covariant derivative from def. A.12. With respect to the representation matrices (A.3), the Dirac operator is then given by

$$\begin{aligned}(\not{D}\sigma)_x &= \sum_{i=1}^{4} e_i \cdot \nabla_i \sigma(x) \\ &= \begin{pmatrix} 0 & -\nabla_1 + \mathrm{i}\nabla_2 + \mathrm{j}\nabla_3 + \mathrm{k}\nabla_4 \\ \nabla_1 + \mathrm{i}\nabla_2 + \mathrm{j}\nabla_3 + \mathrm{k}\nabla_4 & 0 \end{pmatrix} \begin{pmatrix} \sigma_+(x) \\ \sigma_-(x) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\bar{\nabla}^{\mathbb{H}} \\ \nabla^{\mathbb{H}} & 0 \end{pmatrix} \begin{pmatrix} \sigma_+(x) \\ \sigma_-(x) \end{pmatrix} \\ &= \begin{pmatrix} -\bar{\nabla}^{\mathbb{H}}\sigma_-(x) \\ \nabla^{\mathbb{H}}\sigma_+(x) \end{pmatrix},\end{aligned}$$

where $\nabla^{\mathbb{H}} := \nabla_1 + \mathrm{i}\nabla_2 + \mathrm{j}\nabla_3 + \mathrm{k}\nabla_4$ is called the "quaternionic covariant gradient" and its quaternionic conjugate $\bar{\nabla}^{\mathbb{H}} = \nabla_1 - \mathrm{i}\nabla_2 - \mathrm{j}\nabla_3 - \mathrm{k}\nabla_4$.

A.5. The spinor and vector representation in four dimensions

Using the decomposition $\mathrm{C}\ell^0(4) \cong \mathrm{C}\ell(3) \cong \mathbb{H}_+ \oplus \mathbb{H}_-$, the four-dimensional spinor group $\mathrm{Spin}(4) \subset \mathrm{C}\ell^0(4)$ is explicitly given by

$$\mathrm{Spin}(4) = \big\{(q_+, q_-) \in \mathbb{H}_+ \oplus \mathbb{H}_- : |q_+| = |q_-| = 1\big\} = \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$$

according to def. A.7. Let $W := \mathbb{H}$ denote the quaternionic fundamental representation of $\mathrm{Sp}(1)$ by left multiplication. In particular, this representation is isomorphic—however, not in a canonical fashion—to its conjugate $\bar{W}$ when viewed as a complex representation, which in turn is equivalent to right $\mathbb{H}$-multiplication. The four-dimensional spinor representation on $S = \mathbb{H}^2$ is then $\Delta_4 = W^+ \oplus W^-$, i.e.

$$\begin{aligned}\Delta_4 : \mathrm{Spin}(4) \times S &\longrightarrow S \\ \big((q_+, q_-), (v, w)\big) &\mapsto (q_+ v, q_- w).\end{aligned}$$

As an alternative, one frequently finds q_- acting on the negative spinor component by the induced right action, i.e. $\ell_{q_+,q_-}(v, w) = (q_+ v, w\bar{q}_-)$.

This quaternionic formulation can also be used to describe four-dimensional special orthogonal group. For $\mathbb{Z}_2 := \{\pm(1, 1)\} \subset \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-$ it follows

$$\mathrm{SO}(4) = \mathrm{Spin}(4)/\mathbb{Z}_2 = \big\{[q_+, q_-] \in \mathrm{Sp}(1)_+ \times \mathrm{Sp}(1)_-/\mathbb{Z}_2\big\},$$

and the covering map $\mathrm{Spin}(4) \twoheadrightarrow \mathrm{SO}(4)$ is given by the coset projection $(q_+, q_-) \mapsto [q_+, q_-]$. The fundamental representation of $\mathrm{SO}(4)$ on $V := \mathbb{H} \cong \mathbb{R}^4$ can be described by

$$\begin{aligned}\rho_4 : \mathrm{SO}(4) \times V &\longrightarrow V \\ \big([q_+, q_-], v\big) &\mapsto q_+ v \bar{q}_-.\end{aligned}$$

The previous description of $\mathrm{Spin}(4)$, $\mathrm{SO}(4)$ and their representations in terms of quaternions can be applied to manifolds admitting a quaternionic structure.

A.6. Matrix formulation of left quaternionic multiplication

Let $q \in \mathbb{H}$ be a quaternion. With respect to the real basis $1, \mathrm{i}, \mathrm{j}, \mathrm{k}$ it is represented by $q = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d$ for $a, b, c, d \in \mathbb{R}$. Following $\mathbb{H} \cong \mathbb{R}^4$, the components are rearranged into a 4-component column vector

$$q = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \qquad \rightsquigarrow \qquad 1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathrm{i} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathrm{j} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathrm{k} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The effect of a left $\mathbb{H}$-multiplication is then represented by three real 4×4-matrices:

$$\mathrm{i}q = -b + \mathrm{i}a - \mathrm{j}d + \mathrm{k}c = \begin{pmatrix} -b \\ a \\ -d \\ c \end{pmatrix} \qquad \rightsquigarrow \qquad \ell_\mathrm{i} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathrm{j}q = -c + \mathrm{i}d + \mathrm{j}a - \mathrm{k}b = \begin{pmatrix} -c \\ d \\ a \\ -b \end{pmatrix} \qquad \rightsquigarrow \qquad \ell_\mathrm{j} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\mathrm{k}q = -d - \mathrm{i}c + \mathrm{j}b + \mathrm{k}a = \begin{pmatrix} -d \\ -c \\ b \\ a \end{pmatrix} \qquad \rightsquigarrow \qquad \ell_\mathrm{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

which are multiplied from the left. Therefore, the quaternionic covariant gradient operator and its quaternionic conjugate from sec. A.4 are

$$\nabla^{\mathbb{H}} = \mathbb{1}\,\nabla_1 + \ell_\mathrm{i}\nabla_2 + \ell_\mathrm{j}\nabla_3 + \ell_\mathrm{k}\nabla_4 = \begin{pmatrix} \nabla_1 & -\nabla_2 & -\nabla_3 & -\nabla_4 \\ \nabla_2 & \nabla_1 & -\nabla_4 & \nabla_3 \\ \nabla_3 & \nabla_4 & \nabla_1 & -\nabla_2 \\ \nabla_4 & -\nabla_3 & \nabla_2 & \nabla_1 \end{pmatrix}$$

$$\bar{\nabla}^{\mathbb{H}} = \mathbb{1}\,\nabla_1 - \ell_\mathrm{i}\nabla_2 - \ell_\mathrm{j}\nabla_3 - \ell_\mathrm{k}\nabla_4 = \begin{pmatrix} \nabla_1 & \nabla_2 & \nabla_3 & \nabla_4 \\ -\nabla_2 & \nabla_1 & \nabla_4 & -\nabla_3 \\ -\nabla_3 & -\nabla_4 & \nabla_1 & \nabla_2 \\ -\nabla_4 & \nabla_3 & -\nabla_2 & \nabla_1 \end{pmatrix},$$

which are multiplied from the left to a 4-component quaternion column vector.

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