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Semi-Realistic Orbifold Compactification of Heterotic Strings

Mathematical Concepts and Physical Realization

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ABSTRACT. Recent results on orbifold compactifications of the $\mathrm{E}_8 \times \mathrm{E}_8$-heterotic string that lead to 4-dimensional $\mathcal{N} = 1$ supersymmetric effective field theories with three chiral fermion generations subject to the Standard Model gauge group $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ are reviewed. The mathematical formalism is developed in detail, in particular the geometrical properties of gauge theories (i.e. the underlying principal bundle structure) and spinors (Clifford bundles and general spin structures) as required for a proper understanding of toroidal and Calabi-Yau compactifications and gauge symmetry breaking. The Kaluza-Klein process of symmetry breaking is explained in detail and applied to toroidal compactifications. Holonomy and its recurring relevance to the existence of SUSY-generating parallel spinors is emphazised in this context. The exposition ultimately leads to the semi-realistic $\mathbb{Z}_6$-II orbifold model of Buchmüller et al., which is based on the recently introduced concept of local grand unification in the context of orbifold compactification of the heterotic string.

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Φύσις κρύπτεσθαι φιλεῖ.
("Nature is wont to hide herself.")

Heraclitus of Ephesus
"On the Universe"
(Greek Philosopher, c. 535-475 BC)

Contents

CHAPTER 1

Introduction

The early stages of 21st century theoretical physics are marked by peculiarities and curiosities at its very fundamentals. Since the early 1970's elementary particle physics at the microscopic level seems to follow the laws of the quantum-theoretic standard model, whereas Einstein's theory of general relativity provides an ample description of the universe on the large scale. Whenever one of those two pillars of modern theoretical physics is applicable, it gives very precise results, but both seem to be fundamentally incompatible with each other. Both theories give totally different perspectives on the universe that cannot be extended ad hoc to the domain of the respective other. Quantum field theory describes a world governed by intrinsic uncertainties, statistical predictions and discreteness of quantities, which only seems to work in a fixed space-time. General relativity, on the other hand, lays out a universe of smoothly curved dynamic space-time with completely deterministic behavior, but where any attempt to apply the concepts of quantum theory leads to numerous problems of both conceptual and technical nature. While this incompatibility does not seem to affect the understanding of everyday and even laboratory physics, drastic changes are expected in extreme situations where the applicability domains overlap and the fundamental concepts of both theories are needed for a complete understanding of the governing dynamics. For example, the immediate neighborhood of black holes or the supposed space-time singularity encountered in the hot early universe (see [HE73]) are considered as typical situations, where a unified theory of the standard model and general relativity is expected to provide a deeper insight.

The standard model is the quantum-field theoretic union of the Glashow-Weinberg-Salam electroweak model and quantum chromodynamics (QCD), albeit both remain independent of each other aside from affecting the same matter constituents. The former theory evolved from quantum electrodynamics (QED), which was developed by Feynman, Schwinger and Tomonaga in the 1940's as a relativistic local quantum field theory of the electron, whereas QCD is a non-abelian quantum gauge field theory presented by Gell-Mann, Fritzsch and Leutwyler—with Politzer, Wilczek and Gross supplying the crucial discovery of asymptotic freedom—in the early 70's to describe the strong interaction that holds together atomic nuclei. Even today QED as part of the standard model remains the most precise physical theory known.

A serious drawback of the standard model is the high degree of arbitrariness found in its at least 19 free parameters.[a] On the other hand, these parameters have to be fine-tuned to give an universe even remotely similar to ours in terms of interaction strengths, particle lifetimes, matter content, etc. Furthermore, for the chirality asymmetry observed in weak interactions, no deeper theoretical understanding is gained from the standard model, cf. fig. 1.1. This is in part due to the conceptual arbitrariness of the standard model as a quantum gauge field theory, i.e. as a consistent quantum field theory the standard model is not distinguished aside from the fact that it seems to describe all experimentally accessible particle physics data.

[a]Sometimes the number of free parameters in the standard model is given as 18, omitting the strong CP-violating parameter. The number of parameters breaks down as follows: 3 gauge couplings (g_s, e, $\sin\vartheta_\mathrm{W}$), 2 boson masses (m_W, m_Z), 3 lepton masses (m_e, m_μ, m_τ), 6 quark masses (m_u, m_d, m_c, m_s, m_t, m_b), 3 quark mixing angles (ϑ_1, ϑ_2, ϑ_3), 1 weak CP-violating phase δ and 1 strong CP-violating phase Θ. Today it is believed that the neutrinos are not massless, which would enlarge this list by the 3 neutrino masses (m_{ν_e}, m_{ν_μ}, m_{ν_τ}), 3 lepton mixing angles (θ_1, θ_2, θ_3) and 1 CP-violating phase in the leptonic mixing matrix.

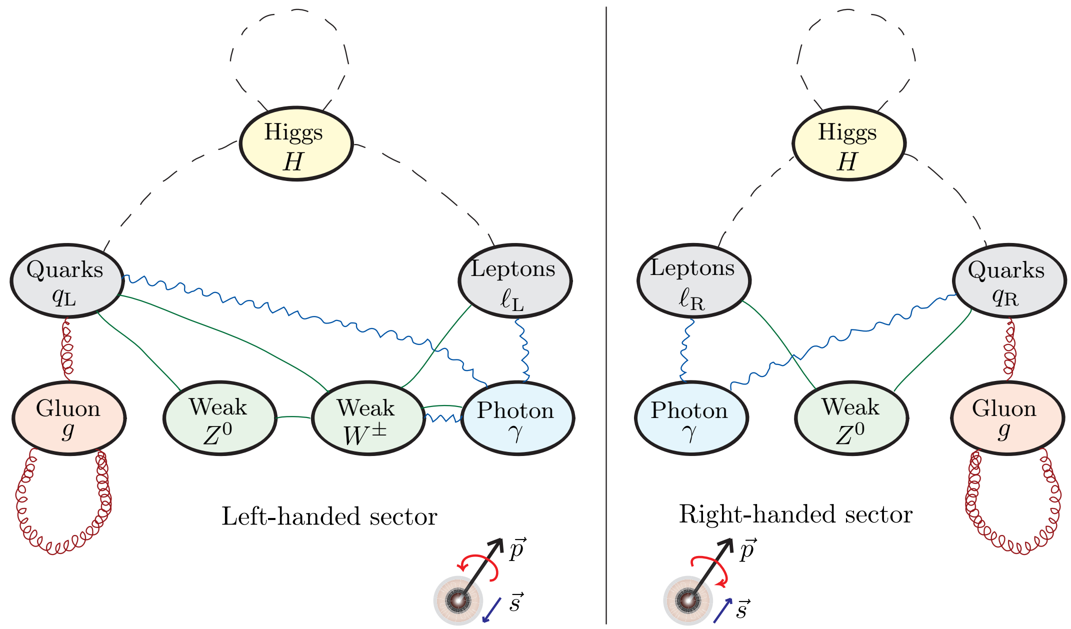


FIGURE 1.1. Interactions and coupling to matter fields in the standard model for the left- and right-handed sector. The upper row contains the undetected Higgs boson, the middle row holds the matter particles and in the lower row the interaction bosons are found. Lines between the particles indicate the interactions between the respective elementary particles.

When first indications of neutrino oscillations and massive neutrinos appeared in the Super-Kamiokande experiment in 1998, this was the first fact not explained in advance by the (unextended) standard model, which had already successfully predicted the existence of the weak interaction bosons $W^\pm$ and Z^0, the gluons g, the charm quark c and the top quark t prior to their actual discovery. Neutrino masses and oscillations can be incorporated into the standard model, but this requires additional free parameters, making the standard model even more unspecified. In contrast, the predicted Higgs boson H, which takes a very important role in the mass-generating spontaneous symmetry breaking process, was not yet observed in experiment. It is hoped that the high-energy particle accelerator LHC, that is scheduled to produce data at the end of 2008, will reach energies high enough to detect it. In essence, despite its successes, the standard model fails to provide a verifiable explanation what massive particles really are.

To account for the abundance of free parameters in the standard model, unified theories (GUTs) are considered, which contain the standard model as a low-energy theory of a conceptually simpler quantum gauge field theory, i.e. the standard model gauge group $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ arises as the subgroup of a unified gauge group (e.g. SU(5), SO(10), E_6, E_7 or E_8) surviving after certain symmetry breaking steps at low energies. This leads to a both phenomenological and conceptual desirable unification of the strong, electromagnetic and weak interaction at very high energies, that provides an ample explanation of certain phase-transition effects expected in the early universe and requires far less parameters. Unfortunately, this unification implies—among other things—a decay of the proton, whose strong suppression puts stringent limits on certain parameters, e.g. lifetime bounds.

Another extension to the standard model utilizes supersymmetry, which assigns to each bosonic interaction particle a fermion partner particle and to each fermionic matter particle an appropriate bosonic counterpart. It has been argued, that these partner particles have eluded experimental detection up to now due to their extremely high mass. This shows that supersymmetry has to be broken strongly, and could furthermore account for the hypothesized dark matter suggested by modern cosmology. One can go on and add further partner particles in the context of extended supersymmetry, but it has been shown that for $\mathcal{N} \geq 2$ the resulting theory is vector-like (i.e. no parity violation), which does not allow for any kind of chirality asymmetry as observed in weak interactions. Thus, the minimal supersymmetric standard model (MSSM) is the extension of the standard model by $\mathcal{N} = 1$ supersymmetry. One of the main motivations to consider supersymmetry is that it provides a solution to the hierarchy problem encountered in the standard model, i.e. the question why fundamental parameters of the theory (particle masses, couplings, etc.) remain—even after renormalization—unaffected by heavy masses. The particular question is, why the weak interaction is 10^{32} times stronger that the gravitational interaction, which can be reformulated to the question, why the renormalized mass of the (still hypothetical) Higgs particle is so much smaller than the Planck mass. Furthermore, supersymmetry allows for a small—but non-vanishing—cosmological constant, which reflects the zero-point energy of the vacuum. However, due to the fact that no kind of supersymmetry was ever observed in experiment it can only play a role as a (spontaneously) broken symmetry. In fact, the presence of unbroken supersymmetry would have cataclysmic effects and would not allow for any kind of chemistry or nuclei bonding.

However, the most serious shortcoming of the standard model is its failure to include general relativity. In 1974 Hawking showed by using semi-classical methods that a curved space-time indeed has noteable quantum effects, which lead to the Hawking radiation supposedly emitted in the ultimate vicinity of a black hole's event horizon, see [Wal94, chp. 7]. Consequently, quantum effects have to be neccessarily taken into account within gravitational physics. From the Einstein field equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$ it follows, that a quantization of general relativity requires to treat the metric tensor field (representing space and time) in the same way ordinary quantum fields are handled. Due to the uncertainty principle it is hoped that the (supposed) space-time singularity at the center of a black hole—as implied by general relativity—would be "smeared out" by a proper quantum mechanical treatment, which would yield a much more intuitive picture of space and time. But in the language of a local quantum field theory the usual procedures to quantize a classical field, like the space-time metric $g_{\mu\nu}$ in this case, are spoiled by uncontrollable infinities arising from self-interactions of the gravitational quanta. This is essentially due to the gravitational coupling constant being dimensionful, as one cannot set all the fundamental constants h, c and G_{N} to unit at the same time. Heuristically, this can be understood by the puzzling facts, that the space-time metric defines the concept of locality (that is crucial for local quantum field theory), but the quantum uncertainty principle gives rise to graviton fluctuations, which in turn again change the dynamic space-time metric. While in the standard model such self-interaction divergences of the non-gravitational interactions can be removed by renormalization techniques, this approach dramatically fails in the case of quantum gravity. Due to this emergence of back-reactions of the metric, local quantum field theory for point particles can only be defined for static flat space-times.

String theory directly attacks this problem by replacing the 0-dimensional point particle concept with a 1-dimensional fundamental object. The extended nature of those so-called strings, albeit extremely small—supposedly of the size of the Planck length $\ell_{\mathrm{pl}} \approx 1.6 \cdot 10^{-35}\,\mathrm{m}$, which is about 10^{-20} times the size of a proton—, naturally smears out the point-like interaction region and provides for a natural cutoff for the problematic self-interactions that arise in every interacting local quantum field theory. Furthermore, the particle spectrum of each string theory contains a gravitational quantum called the graviton (massless spin-2 particle), such that a consistent quantum theory of gravitation is obtained in a rather natural way. But due to the minuscule effects of gravitational corrections to particle physics in the laboratory, as well as the huge gap between the energies needed to actually probe the extended string nature (of order $10^{16}\,\mathrm{GeV}$) and the energies available in the most modern accelerator (of order $10^{3}\,\mathrm{GeV}$), there is no direct experimental data to guide the physical intuition in string theory. This is in

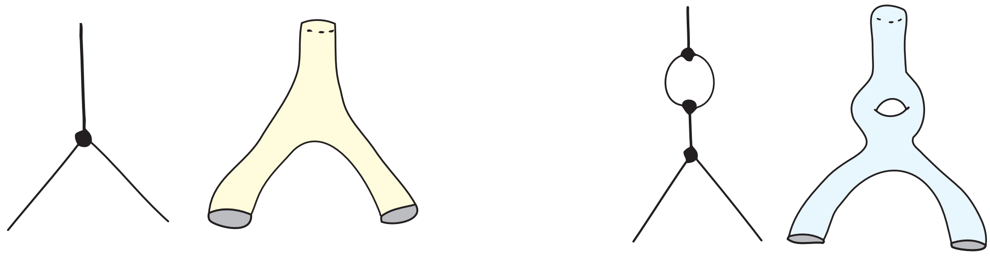


FIGURE 1.2. Comparison of an interaction vertex and the first loop correction in a local quantum field theory and closed string theory.

contrast to the situation in the 60's, when the standard model was still in development. An abundance of seemingly unrelated elementary particles provided plenty of experimental data that had to be explained in a comprehensive theory. From this point of view, there is no data available today to guide string theorists in ways similar to the creators of the standard model.

On the other hand, a consistent relativistic quantum theory of 1-dimensional objects is strongly constrained by consistency conditions. In a certain sense, mathematical elegance and consistency take the place of physical input due to the lack of physical data—which is of course a potentially dangerous route to follow when searching for a unified physical theory, as nature might not turn out to appreciate mathematical beauty. However, history shows that all successful theories of physics sooner or later turned out to be governed by rather neat mathematics: Riemannian geometry and geodesics are the fundamentals of general relativity, whereas quantum mechanics builds on functional analysis and operator algebras. The crucial property of string theory seems to be, that it rather naturally leads to the central principles of modern physics, such as general relativity, quantum mechanics and gauge theory, without prior assumption of those. Even supersymmetry, albeit not established in experimental data, is provided (or rather required) by any potentially realistic string theory, which gives the impression that strings really build up a uniquely constrained theory.

On the other hand, the mathematical and physical consistency of string theory leads to the necessity of a higher dimensional space-time. Instead of the usual 3+1 dimensions of the standard model, superstring theory can only be formulated in 9+1 dimensions (or 25+1 dimensions for the purely bosonic theory, which is, however, inconsistent due to the presence of a tachyon). Higher-dimensional space-time was not unknown in theoretical physics before the advent of the string paradigma. As early as 1921 Kaluza proposed a 5-dimensional extension of general relativity to include electrodynamics, which was developed further by Klein. Their general results in the context of compact extra dimensions are still used today to account for the 6 extra spatial dimensions that are required by superstring theory in addition to the perceived 3+1 space-time dimensions. Kaluza-Klein compactification relates the geometry of the compactification space with the coupling constants and broken gauge groups of the compactified theory in an elegant way. Unfortunately, even after applying all known constraints, the abundance of consistent 6-dimensional compactification geometries brings back an arbitrariness into the resulting effective 4-dimensional theories that resembles the arbitrariness of the standard model as a quantum field theory and its free parameters. This variety of possible effective string theories is called the string landscape and offers profound new insights into the nature of the universe (or perhaps multiverse).

The central aim of this work is to present a concise review and summary of the different ways to compactify 6 of the 10 space-time dimensions required for a consistent definition of the heterotic superstring, such that an effective 4d low-energy field theory with rather realistic properties is obtained. The line of reasoning will be the following: The simplest approach would be 6d toroidal compactification. But it is readily shown that this inevitably leads to 4d

$\mathcal{N} = 4$ supersymmetry in the effective theory, which does not allow for a chiral asymmetry in the interactions of the fermionic sector as found in the experimentally well-established standard model—only $\mathcal{N} = 1$ supersymmetric theories contain such chiral fermions. In the more general approach of a compact 6d Riemannian manifold, it turns out that the metric must have SU(3)-holonomy in order to provide 4d $\mathcal{N} = 1$ supersymmetry in the effective field theory. The corresponding structure of the compactified space-time dimensions is that of a Calabi-Yau manifold. The mathematical implications of this condition are discussed in great detail. Unfortunately, due to the highly non-linear nature of the corresponding partial differential equations imposed on the metric, no non-trivial such Calabi-Yau manifolds are known explicitly. Topological properties allow to access some of the implied phenomenological properties of the effective 4d field theory, but the calculational possibilities are rather limited.

An alternative is compactification on certain singular spaces constructed from 6d tori, called (toroidal) orbifolds. The singularities arise from fixpoints of a group's action, i.e. an orbifold is (locally) the quotient space M/G of a smooth manifold and a finite group acting on it. This approach retains the same level of calculability as found in toroidal compactification, but also breaks the supersymmetry to $\mathcal{N} = 1$, which is again due to SU(3)-holonomy in a disguised form. Furthermore, the geometry of such orbifold spaces allow for so-called twisted closed strings, that are wrapped around a singularity. Specifically, the Yang-Mills gauge group $\mathrm{E}_8 \times \mathrm{E}_8$ is broken independently for each singularity and may be further modified by introducing Wilson lines. The intersection of all those local gauge groups gives the gauge group of the 4d effective field theory. For a suitable choice of the orbifold geometry and Wilson lines, the local gauge group at each singularity contains a copy of a GUT group like SO(10), such that the intersection of these yields the standard model gauge group $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$. In the context of further structures, this allows to construct a semi-realistic effective 4d field theory, that closely resembles the 4d $\mathcal{N} = 1$ minimal supersymmetric extension of the standard model. This is explained for the particular example $T^6/\mathbb{Z}_6$-II recently constructed by Buchmüller, Hamaguchi, Lebedev and Ratz.

Overview of chapters

In chap. 2 gauge theory is introduced in a purely mathematical fashion. The entire chapter stresses the fact, that the objects physicist's tend to call "fields" are in fact sections of certain bundles. In the particular case of gauge potentials, it turns out that gauge and field strengths are directly related to the curvature of connections on certain bundles. In this process, the entire differential geometry of principal bundles is developed, which is the main approach taken in mathematics nowadays. Finally, contact is made with Riemannian geometry and the chapter closes with a short example how the electromagnetic potential A_μ and its field strength $F_{\mu\nu}$ should be understood.

Chap. 3 is a rather technical chapter, were the basics of homology and cohomology theory are summarized. In order to make sense of the numerous abstract constructions, the de Rham cohomology is discussed and applied to existence questions of potentials of rotation-free vector fields, etc. It becomes obvious, that quite a number of such properties strongly depend on the topology of the underlying space, which is encoded in the Betti and Hodge numbers. The second half of the chapter introduces the important subject of characteristic classes—a mathematical tool to measure the geometric properties of bundles. The approach is purely algebraic and essentially only lists the important properties of Stiefel-Whitney, Euler and Chern classes, which will become important in the next chapter.

Spinors in physics are a somewhat strange subject, as many important mathematical properties are often neglected. Chap. 4 starts with a complete introduction to real and complex Clifford algebras. In particular, it is stressed that from a mathematical point of view spinors are not "double-valued representations" of the Lorentz group—as it is often explained in the physical literature—but stem from representations of certain covering Lie groups. Since the Lorentz group consists of more than one connection component, on has to distinguish between "pinor" representations of the full Lorentz group and "spinor" representations of its proper orthochronous subgroup. The splitting of pinor in spinor representations—which provides the chirality splitting of the Dirac spinor in physics—is explained in detail. Those concepts are brought into contact with bundles, which gives rise to the notions of spin bundles and spin structures. The existence of such structures strongly depends on the topology of the underlying base space and is measured by the vanishing of certain characteristic classes. The chapter closes with a short summary how the concepts common in physics should be treated from a more mathematical perspective.

With this additional material, chap. 5 continues where chap. 2 ended. It starts with a brief review of Riemannian geometry and introduces the tensor notation used in physics. The entire chapter focuses on the notion of holonomy, i.e. how a tangent vector is changed under parallel transport around a closed loop. It turns out that all Riemannian manifolds can be classified with respect to their holonomy group. The particular class of Ricci-flat manifolds allows for the existence of non-zero global covariantly constant spinors. Those are of utmost importance in the succeeding chapters, as such spinors serve as generators of the supersymmetry. Finally, the Dirac operator is introduced, which is another linking object between Riemannian and spin geometry.

The following chapters are devoted to the physical application of the mathematical concepts and theory. Chap. 6 summarizes the concepts of supersymmetry. Based on the Poincaré algebra, the Minkowski space is constructed as the natural space invariant under the corresponding Poincaré group. Next, the notion of a Lie algebra is extended to include anticommuting skew-symmetric brackets. Following the same lines of reasoning as before, this yields the Poincaré superalgebra and supergroup, with the Minkowski superspace-time being the underlying space. Supersymmetry transformations and irreducible representations of the Poincaré supergroup—called supermultiplets—are investigated for several special cases of interest. All this is carried out using the conceptually simpler superspace formalism, which is brought into contact with the component field approach afterward. The chapter closes with a review of the vielbein formalism and a short summery of supergravity.

In chap. 7 the concept of string is introduced and developed up to the notion of the heterotic string. Following the usual way of presentation, the closed bosonic string is introduced at first. Much space is spent to detail the numerous symmetries and possible extensions one may add to the basic construction. In general, the chapter highlights several aspects of string theory, e.g. a section is spent on the problem of the background depending formulation, another on the finiteness of superstring theories. After the introduction of supersymmetric strings, the mode expansion and the quantization process are summarized. The heterotic string is then constructed by adjoining bosonic strings and superstrings, such that closed-string Yang-Mills states arise from the toroidal compactification of the unmatched bosonic components. After listing the massless heterotic particle spectrum, the low-energy effective supergravity approximation is discussed and supported by a brief review of the famous Green-Schwarz anomaly cancellation procedure.

Chap. 8 explains the Kaluza-Klein mechanism of dimensional compactification. This is then carried out for the toroidal compactification of the heterotic string, yielding a highly unrealistic 4d $\mathcal{N} = 4$ supersymmetric effective theory with no chiral matter and very large gauge group. In order to reduce the amount of supersymmetry, conditions are derived from the SUSY transformation behavior of the SUGRA approximation, which is carried out in great detail. Those conditions finally require the existence of covariantly constant global spinors, ultimately forcing the internal space to be a Calabi-Yau manifold. The phenomenological properties of such Calabi-Yau compactifications are then summarized and the chapter closes with a short discussion of the shortcomings of both approaches.

In order to achieve the same phenomenological success as with the Calabi-Yau manifolds, but also the calculability of the toroidal approach, orbifolds are finally introduced in chap. 9. After a general, mathematical treatment of the subject—defining the notions of holonomy for such singular spaces—toroidal Calabi-Yau orbifolds are investigated. The general constructions are applied to the explicit examples $T^6/\mathbb{Z}_3$ and $T^6/\mathbb{Z}_2$, which have conical and toric singularities. Finally, orbifold compactifications of the $\mathrm{E}_8 \times \mathrm{E}_8$-heterotic string are developed. The modification of the closed-string boundary conditions, the embedding of the space group action (including Wilson lines) and the required consistency conditions—guaranteeing modular invariance and an anomaly-free string—are investigated in detail. Next, the remaining particle spectrum (both for the twisted and untwisted sector) and the projection conditions are summarized. The discussion of the Hilbert spaces for different sectors finishes the generalities on orbifold compactifications.

Chap. 10 starts with a summary of grand unified theories (GUTs), focusing on the classic SU(5) and SO(10) models. As shown, the latter has rather pleasant conceptual properties, which are unfortunately spoiled by problems related to the rapid proton decay, etc. Such issues can be avoided in local GUTs, which are based on orbifold compactification of string theory. Essentially, each orbifold singularity potentially may break the original $\mathrm{E}_8 \times \mathrm{E}_8$-symmetry in a different manner, such that the compactified theories gauge group is the intersection of all those local groups. This provides the basis for the $\mathbb{Z}_6$-II model recently presented in [BHLR06b], yielding the semi-realistic MSSM along with many attractive phenomenological properties. Due to certain degrees of freedom found in this construction, the notion of an "orbifold landscape" arises. The chapter closes with a few remarks on the ongoing research in this direction.

Chap. 11 presents an outlook on several recent developments in string compactifications, i.e. flux compactifications, the general string landscape and connections to the focused orbifold compactifications.

The entire text is formulated in a rather rigorous mathematical language, which makes use of many notions mathematicians would consider to be elementary, but which are not familiar to most physicists. A rather lengthy introduction to nearly all those techniques in presented in a concise manner in the appendix. App. A starts with groups, elementary topology and manifolds. In further sections bundles are introduced, as well as vector fields. The exterior algebra provides the means to define differential forms, which are used throughout the entire main text. The chapter closes with a review of Lie groups and algebras, accompanied by a few words on their representation theory. In app. B more advanced topics from representation theory are introduced, like weights and root systems. Furthermore, the rather elegant denotation of root systems in terms of Dynkin diagrams is used. The rest of the chapter lists the properties of most groups used in the main text, particularly the root systems, whose simple roots serve as torus lattices.

Any reader unfamiliar to strict mathematics is strongly advised to start reading app. A. While working through the main text, the appendix should be used for general reference, as its contents are considered to be elementary and therefore are not referenced.

Part I

Geometry of Physical Objects

CHAPTER 2

Differential Geometry and Gauge Theory

Quantum gauge field theory became the most important building block of fundamental physics, when t'Hooft proved the standard model to be renormalizable to all perturbative orders in 1971. To understand the physical properties of Calabi-Yau compactification and the process of gauge symmetry breaking properly, the mathematical details of differential geometry have to be developed beyond the common local coordinate dependent notions usually encountered in physics. Most of this material is reviewed in greater detail in [Gre97, chp. 2] and various mathematical textbooks, e.g. classics like [Mil65] or [GP74]. Differential geometry is introduced via the principal bundle approach, which is the basis of mathematical gauge theory. The general references for the respective sections are the classical—albeit quite dated—textbooks [KN63] and [KN69] for more advanced topics. A somewhat more accessible introduction for physicist is presented in [Nab97, chaps. 3-5]. In particular, the later sections make heavy usage of the first two lectures on gauge theory compiled in [Fig06b], wherein the rather exhaustive article [DV80] is summarized and updated.

2.1. Principal fiber bundles and transition functions

Bundles are manifolds which locally take the form of a product space. It consists of a total space E, a fiber F and a base space M, all of which are smooth manifolds with possible additional structure. A projection mapping $\pi : E \twoheadrightarrow M$ has a preimage $\pi^{-1}(p) \cong F$ for any point $p \in M$, see sec. A.7 for more details. In case of a real vector bundle $E \xrightarrow{\pi} M$ of rank n, every point $p \in M$ has a neighborhood $U \subset M$, such that the bundle is locally diffeomorphic to $U \times \mathbb{R}^n$—however, in general this does not hold globally.[a]

In physics one often encounters quantities which appear to be sections of trivial vector bundles over space-time, an important example being the potential A_μ in classical electrodynamics. However, despite appearing to be just an ordinary section in a vector bundle, this particular object has additional structure in the form of U(1)-gauge transformations, which must be taken into consideration.

This is achieved, if the bundle's fiber is chosen to be a Lie group G. Let (P, M, G, π) be a fiber bundle, then it is called a **principal G-bundle** if it has a continuous right action $P \times G \longrightarrow P$ of G on the total space such that the Lie group G preserves the fibers of P and acts freely and transitively[b] on them. From the definition it follows that the orbits of the G-action are precisely the fibers of $P \xrightarrow{\pi} M$ and the orbit space P/G is homeomorphic to the base space M. Let $\phi_\alpha : P|_{U_\alpha} \xrightarrow{\approx} U_\alpha \times G$ be a local trivialization of the principal G-bundle over $U_\alpha \subset M$ as introduced in app. A for general fiber bundles, i.e. the diffeomorphism that makes the bundle's local product structure manifest. After introducing a "fiber trivialization function" $g_\alpha : U_\alpha \longrightarrow G$, such that

$$\phi_\alpha(p) = \big(\pi(p), g_\alpha(p)\big) \in U_\alpha \times G$$

[a]All of this is covered in much greater detail in the appendix, in particular see sec. A.7. A reader unaccustomed to those terms is strongly advised to first reading app. A.

[b]An action $G \times M \longrightarrow M$ of a group G on the space M is called **transitive** if for any given pair $x, y \in M$ of points there exists an group element $g \in G$ such that $gx = y$. The action is called **free** if for any two different group elements $g \neq h$ each space point $x \in M$ is mapped to a different point $gx \neq hx$. Equivalently, given a free action, then $gx = x$ for some $x \in M$ directly implies $g = e$.

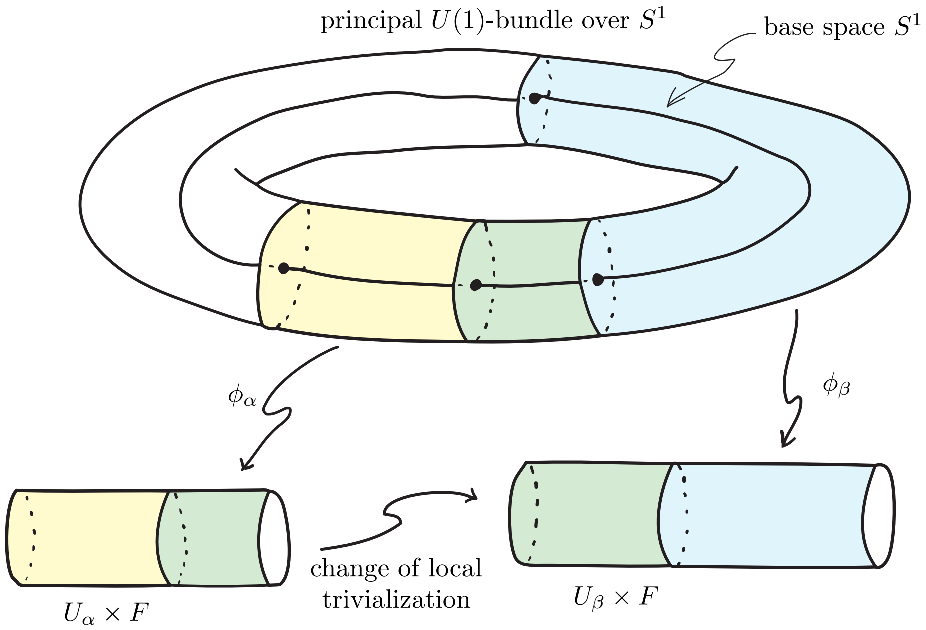


FIGURE 2.1. A trivial principal G-bundle over the circle.

holds with respect to the local trivialization ϕ_α, the structure of a principal G-bundle implies ***G*-equivariance** of g_α, which means $g_\alpha(pg) = g_\alpha(p)g$ for all $p \in P$ and $g \in G$.

Consider two local trivializations (ϕ_α, U_α) and (ϕ_β, U_β) with nonempty overlap $U_{\alpha\beta} := U_\alpha \cap U_\beta$. On this overlap there are two ways of trivializing the bundle, cf. fig. 2.1:

$$\begin{tikzcd}
U_{\alpha\beta} \times G \arrow[rr, dashed, bend left, "(p,g)\mapsto(p,g_{\beta\alpha}(p)g)", "\phi_\beta\circ\phi_\alpha^{-1}"'] \arrow[dr, "\mathrm{pr}_1"'] & P|_{U_{\alpha\beta}} \arrow[l, "\phi_\alpha", "\approx"'] \arrow[r, "\phi_\beta"', "\approx"] \arrow[d, "\pi"] & U_{\alpha\beta} \times G \arrow[dl, "\mathrm{pr}_1"] \\
& U_{\alpha\beta} &
\end{tikzcd}$$

The function $g_{\beta\alpha} : U_{\alpha\beta} \longrightarrow G$ is the **transition function** with respect to the local trivializations ϕ_α and ϕ_β, and describes the point-wise change of the fiber when passing from one trivialization to another (see dashed arrow in the diagram above). Moreover, on overlaps the transition functions satisfy the **cocycle conditions**[c]

$$\begin{aligned}
&g_{\alpha\alpha} = \mathrm{Id} && \text{(on any patch } U_\alpha\text{)} \\
&g_{\alpha\beta} g_{\beta\alpha} = \mathrm{Id} && \text{(on twofold overlaps } U_{\alpha\beta}\text{)} \\
&g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathrm{Id}\,. && \text{(on threefold overlaps } U_{\alpha\beta\gamma}\text{)}
\end{aligned}$$

Note that the entire collection of transition functions encodes the global topology of the principal G-bundle $P \xrightarrow{\pi} M$. Given an open cover $\{U_\alpha\}$, i.e. a collection of open subsets $U_\alpha \subset M$ satisfying $\bigcup_\alpha U_\alpha = M$, and suitable transition functions $\{g_{\alpha\beta}\}$ obeying the cocycle conditions, the bundle is reconstructed via

$$P = \coprod_\alpha (U_\alpha \times G) \big/ \sim$$

[c]The name "cocycle conditions" comes from certain constructions of the cocycles encountered in Čech cohomology, see [BT82, §10].

where the equivalence relation "$\sim$" is defined by the identifications $(p,g) \sim \big(p, g_{\alpha\beta}(p)g\big)$ that relate the different bundle patches.[d] One of the first textbooks on the subject, [Ste51], almost exclusively uses this description of fiber bundles. An updated and extensive reference on fiber bundle theory is found in [Hus98], and a physically motivated example is discussed in sec. 2.9.

It is important to realize, that non-triviality of principal G-bundles is a much stronger notion than non-triviality of vector bundles. Whereas in vector bundles one can always take the global constant **zero section** (which is usually understood as a canonical embedding of the base space M in the bundle's total space E), a global section in a principal G-bundle only exists if the bundle is trivial. Of course, local sections are always possible by construction.

The linear structure and behavior of vector bundles implies, that on a change of trivialization the group $\mathrm{GL}_k(F)$ acts on the trivialized fiber, where k is the dimension of the attached vector spaces over the field F. Thus, any vector bundle can be understood as a principal $\mathrm{GL}_k(F)$-bundle, called the **frame bundle**. Conversely, one can construct vector bundles from a given principal G-bundle, which is explained in the next section. Thus, principal G-bundles are much more general than vector bundles. [Joy00, chp. 2] provides a concise account on frame bundles.

2.2. Associated bundles and bundle-valued forms

There is a way to replace the Lie group fiber of a principal G-bundle with another, almost arbitrary fiber. Let $P \xrightarrow{\pi} M$ be a principal G-bundle, F a differentiable manifold (the new fiber that is to be attached) and $\rho : G \longrightarrow \mathrm{Diff}(F)$ a smooth effective left action of G on the space F. Then define a right action of G on the product space $P \times F$ via

$$\begin{aligned} \tilde{\rho} : (P \times F) \times G &\longrightarrow P \times F \\ (p,f)g &:= \big(pg, \rho_{g^{-1}}(f)\big) \end{aligned} .$$

Using this action, the orbit space $P \times_\rho F := (P \times F)/G$ is well-defined and called the **associated bundle** with fiber F and structure group G. Further mathematical details on this construction are found in [KN63, §I.5], for example.

This somewhat obscure definition is just the correct mathematical definition for the replacement of the Lie group fiber G with another fiber F in a way compatible with the bundles global structure. Usually F is a finite-dimensional vector space V and ρ a linear representation of the bundle's Lie group G on V. As mentioned before, there is a one-to-one correspondence between vector bundles and principal GL_k-bundles via the notion of frame bundles and associated vector bundles.

Two very important associated bundles are the **associated adjoint bundles** of a principal G-bundle. Using the conjugation mapping $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(G)$ and the Lie group's adjoint representation $\mathrm{ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$, as introduced in the sec. A.19, define

$$\begin{aligned} \mathrm{Ad}\, P &:= P \times_{\mathrm{Ad}} G && \text{(associated fiber bundle)}, \\ \mathrm{ad}\, P &:= P \times_{\mathrm{ad}} \mathfrak{g} && \text{(associated vector bundle)} \end{aligned}$$

as the two canonical bundles associated to any principal G-bundle. In the following sections the latter of these two bundles, $\mathrm{ad}\, P \xrightarrow{\tilde{\pi}} M$, will be of much importance, and it will be referred to as the **associated Lie algebra bundle**.

Consider the trivial product bundle $M \times \mathfrak{g} \xrightarrow{\mathrm{pr}_1} M$ associated with the Lie algebra $\mathfrak{g}$. Then $\Omega^k_M(\mathfrak{g})$ denotes **Lie algebra-valued differential forms** with respect to this product bundle, which for a function $\omega_\mathfrak{g} : M \longrightarrow \mathfrak{g}$ and an ordinary k-form $\tilde{\omega} \in \Omega^k_M$ on M can be expressed as

$$\omega = \tilde{\omega} \otimes \omega_\mathfrak{g}.$$

[d] These constructions can be understood in a much more general sense by introducing the notion of sheaves, see [Bre67]. The topology of the base space is encoded in the Čech cohomology groups, which are constructed as inductive limits of coverings after introducing a certain refinement ordering relation on the set of coverings. In [HKK+03, chp. 2] some of these notions are introduced in a way that is more accessible for physicists.

The natural product operator for Lie algebra-valued forms is a combination of the ordinary wedge product and the Lie bracket via

$$\begin{aligned}[.,.] : \Omega^k_M(\mathfrak{g}) \otimes \Omega^l_M(\mathfrak{g}) &\longrightarrow \Omega^{k+l}_M(\mathfrak{g}) \\ \omega \otimes \eta \mapsto [\omega, \eta] &:= \tilde{\omega} \wedge \tilde{\eta} \otimes [\omega_{\mathfrak{g}}, \eta_{\mathfrak{g}}]_{\mathfrak{g}}\end{aligned},$$

where $[.,.]_{\mathfrak{g}}$ refers to the Lie bracket of the Lie algebra $\mathfrak{g}$.

2.3. Connections

In simple terms, connections are what the physicist calls the "gauge" of a gauge field, which is in fact a principal G-bundle with G being the corresponding gauge group. This will be elaborated further in the appropriate sec. 2.6.

Let $P \xrightarrow{\pi} M$ be a principal G-bundle and $p \in \pi^{-1}(x)$ a point within the fiber over $x \in M$. The tangent space T_pP contains a subspace tangent to the fiber $\pi^{-1}(x)$, called the **vertical subspace** $V_p = \ker \pi_*$, where $\pi_* = \mathrm{T}\pi : \mathrm{T}P \longrightarrow \mathrm{T}M$ is the point-wise tangent mapping of the bundle projection. Vector fields $W \in \mathfrak{X}(P)$, where $\mathfrak{X}(P) = \Gamma(\mathrm{T}P)$ is the space of sections of the tangent bundle (i.e. vector fields), on the principal bundle's total space are called **vertical** if $V(p)$ is an element of the respective vertical subspace at each point $p \in P$.

Given the vertical subspace, which is provided by the bundle structure itself, there is no natural complement to V_p within T_pP. It is important to remember, that there is no metric specified yet (as it would be the case in Riemannian geometry) which could provide such a complement. A **connection** H on P exactly provides such a structure, as it is a smooth choice of **horizontal** subspaces $H_p \subset \mathrm{T}_pP$ such that for all $p \in P$

$$\mathrm{T}_pP = V_p \oplus H_p \qquad \text{and} \qquad (r_g)_* H_p = H_{pg},$$

where $(r_g)_*$ is the push-forward of the group's right-multiplication operation, which implies right-invariance for H. In summary, the essential idea of connections is to provide—without reference to any metric—a distinction between vertical and horizontal tangent vectors on P by specifying a subbundle $H \subset \mathrm{T}P$, which is called a **distribution**.[e]

The action of the Lie group G on P defines a natural mapping $\xi : \mathfrak{g} \longrightarrow \mathfrak{X}(P)$, assigning to each Lie algebra element $X \in \mathfrak{g}$ the **fundamental vector field** $\xi(X)$ whose value at p is given by

$$\xi_p(X) = \frac{\mathrm{d}}{\mathrm{d}t} \left(p\, \mathrm{e}^{tX}\right)\Big|_{t=0}.$$

It is important to notice that $\pi_* \xi_p(X) = 0$ for all $p \in P$, thus the fundamental vector field is vertical everywhere.

While the above definition of connections is geometrically very suggestive, it it quite cumbersome to work with. Consider the projection $\mathrm{T}_pP \twoheadrightarrow V_p$ onto the vertical vectors. There is a natural isomorphism $V_p \cong \mathfrak{g}$ provided at each point $p \in P$ by the inverse mapping of the fundamental vector field $\xi_p : \mathfrak{g} \longrightarrow V_p$. A **connection 1-form** of a connection $H \subset \mathrm{T}P$ is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1_P(\mathfrak{g})$, which for any vector field $V \in \mathfrak{X}(P)$ is defined through

$$\omega(V) = \begin{cases} X & : \ V = \xi(X) \text{ is vertical, where } X \in \mathfrak{g} \\ 0 & : \ V \text{ is horizontal} \end{cases}$$

under the usage of linearity, i.e. the differential form ω is the projection onto the vertical subspace under the point-wise identification $V_p \cong \mathfrak{g}$. In general, a form on P is called **horizontal** if it is vanishes on vertical vectors. Furthermore, the connection 1-form obeys

$$(r_g)^*\omega = \mathrm{ad}_{g^{-1}}(\omega),$$

which will also be called G-equivariance later on. Conversely, for any $\mathfrak{g}$-valued 1-form that satisfies both this identity and $\omega\big(\xi(X)\big) = X$ the associated distribution $H = \ker \omega \subset \mathrm{T}P$ defines a connection on $P \xrightarrow{\pi} M$. Extensive mathematical details of these definitions are

[e] The term "distribution" herein is not to be confused with the equal notion used for generalized functions or for probability distributions used in stochastic.

provided in [KN63, §II.1], whereas [DV80, §2.C] is a rather abridged summary of the most important facts.

The existence of connections in general is a rather nontrivial result, see [KN63, §II.2]. More importantly, the **space of connections** $\mathscr{A}$ is an infinite-dimensional affine space, i.e. a vector space plus a reference connection ω_0. Consider two connection 1-forms $\omega, \omega' \in \Omega^1_P(\mathfrak{g})$ for two connections $H, H' \subset \mathrm{T}P$, then their difference $\omega - \omega' \in \Omega^1_P(\mathfrak{g})$ is horizontal and still G-invariant. Therefore, any connection can be represented as $\omega = \omega_0 + \alpha$ for a suitable $\alpha \in \Omega^1_M(\operatorname{ad} P)$, which makes the affine structure of the space of connections $\mathscr{A}$ manifest.

2.4. Basic forms

To understand the intricate relation between forms on P and forms on M, as used to describe the space of connections $\mathscr{A}$, some technicalities have to be investigated.

Let $P \xrightarrow{\pi} M$ be a principal G-bundle with a connection $H \subset \mathrm{T}P$, then one can define a **horizontal projection** $h : \mathrm{T}P \longrightarrow \mathrm{T}P$ onto the horizontal subbundle given by the connection. Furthermore, let $h^* : \mathrm{T}^*P \longrightarrow \mathrm{T}^*P$ denote the dual map, that is

$$(h^*\alpha)(V_1, \dots, V_k) = \alpha(hV_1, \dots, hV_k)$$

as for pull-backs of higher differential forms. It is important to realize, that despite the notation, the mapping h^* is not the pull-back by a differentiable function. This implies in particular, that h^* will not commute with the exterior differentiation, i.e. $\mathrm{d}(h^*\alpha) \neq h^*\mathrm{d}\alpha$. A horizontal k-form $\alpha \in \Omega^k_P$ of $P \xrightarrow{\pi} M$ is called **basic** if it is also G-invariant. In fact, α is basic if and only if $\alpha = \pi^*\tilde{\alpha}$ for some $\tilde{\alpha} \in \Omega^k_M$, i.e. a basic k-form on P is the pull-back of a k-form on M via the bundle projection. This concept will now be generalized to bundle-valued differential forms as required for the language of gauge theory.

First, let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a linear representation of the bundle's Lie group G on the vector space V, then a V-valued k-form $\beta \in \Omega^k_P(V)$ is called ***G*-invariant** if

$$(r_g)^*\beta = \rho_{g^{-1}}(\beta)$$

holds for all $g \in G$ and horizontal if $h^*\beta = \beta$, as before. This G-equivariance was already encountered in sec. 2.3, where it appears in the special case of $\rho = \operatorname{ad}$ for the connection 1-form. If β is both G-equivariant and horizontal, it is called **basic**. Define

$$\Omega^k_P(V)^\flat := \Omega^k_P(V)^\#_G = \left\{ \alpha \in \Omega^k_P(V) \text{ with } \begin{array}{ll} h^*\alpha = \alpha & \text{(horizontal)} \\ (r_g)^*\alpha = \rho_{g^{-1}}(\alpha) & (G\text{-equivariant}) \end{array} \right\}$$

to be the **space of basic *V*-valued *k*-forms**, where "$\flat$" indicates basic forms, "G" stands for G-equivariant forms and "$\#$" refers to horizontal forms.[f]

These (generalized) basic forms are in bijective correspondence with $(P \times_\rho V)$-valued forms on M. The isomorphism can be described locally as follows: First note, that every local trivialization $\phi_\alpha : P|_{U_\alpha} \longrightarrow U_\alpha \times G$ of a principal G-bundle provides a **canonical local section** $s_\alpha : U_\alpha \longrightarrow P|_{U_\alpha}$, such that for every $x \in U_\alpha$

$$\phi_\alpha \circ s_\alpha(x) = (x, e)$$

holds, where $e \in G$ refers to the group's unit element. That is, s_α is constant unity with respect to its appearance under the local trivialization ϕ_α. For any basic k-form $\tilde{\eta} \in \Omega^k_P(V)^\flat$ define a local pull-back patch

$$\eta_\alpha := s^*_\alpha \tilde{\eta} \in \Omega^k_{U_\alpha}(V)$$

on U_α. On nonempty overlaps the local patches are related via $\eta_\alpha = \rho_{g_{\alpha\beta}}(\eta_\beta)$, thus the collection $\{\eta_\alpha\}$ of all pull-backs constitutes a k-form $\eta \in \Omega^k_M(P \times_G V)$. Conversely, if $\eta_\alpha \in \Omega^k_{U_\alpha}(V)$ is a local patch of a given k-form $\eta \in \Omega^k_M(P \times_G V)$, then define

$$\tilde{\eta}_\alpha := \rho_{g_\alpha^{-1}}(\pi^*\eta_\alpha) \in \Omega^k_{P|_{U_\alpha}}(V).$$

[f] This particular choice of notation is not conventionally used in the literature.

Again, the collection $\{\tilde{\eta}_\alpha\}$ constitutes a k-form $\tilde{\eta} \in \Omega^k_P(V)$, which completes the description of the isomorphism. Finally, this can be summarized graphically as

$$\begin{tikzcd} \Omega^k_P(V)^\flat \arrow[rr, bend left, "\tilde{\eta} \mapsto s_\alpha^* \tilde{\eta} = \eta_\alpha"] & \cong & \Omega^k_M(P \times_G V) \arrow[ll, bend left, "\tilde{\eta}_\alpha = \rho_{g_\alpha^{-1}}(\pi^* \eta_\alpha) \mapsfrom \eta_\alpha"] \end{tikzcd} . \tag{2.1}$$

In particular, this provides $\Omega^k_P(\mathfrak{g})^\flat \cong \Omega^k_M(\operatorname{ad} P)$ when using $V = \mathfrak{g}$ and $\rho = \mathrm{Ad}$, which is the essential translation tool between physical and mathematical notions of gauge field theory. This was already used in the description of the affine structure of $\mathscr{A}$.

2.5. Curvature and the exterior covariant derivative

Let $\omega \in \Omega^1_P(\mathfrak{g})$ be the connection 1-form associated to the horizontal distribution $H \subset \mathrm{T}P$, then the 2-form $\Omega := h^* \mathrm{d}\omega \in \Omega^2_P(\mathfrak{g})$ is called the **curvature 2-form**.[g] Consider

$$\begin{aligned} \Omega(V, W) = \mathrm{d}\omega(hV, hW) &= (hV)\underbrace{\omega(hW)}_{0} - (hW)\underbrace{\omega(hV)}_{0} - \omega\big([hV, hW]\big) \\ &= -\omega\big([hV, hW]\big), \end{aligned}$$

where the additional terms vanish since $h^*\omega = 0$ by construction, then $\Omega(V, W) = 0$ holds if and only if the Lie bracket of vector fields $[hV, hW]$ is horizontal. A connection with vanishing curvature is called **flat**. The important interpretation of this equality is that the curvature of a connection measures the failure of integrability of the horizontal distribution $H \subset \mathrm{T}P$, i.e. the deviation of the horizontal vector fields to form a Lie algebra. Furthermore, there is the equality

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega],$$

called the (second) **structure equation**, which is much more useful for actual calculations. A proof for this statement is found in [KN63, thm. II.5.2]. From this the **Bianchi identity** follows by simple calculation:

$$h^* \mathrm{d}\Omega = 0.$$

The specification of a connection allows to define a derivative on sections of vector bundles associated to principal G-bundles, as will be shown in sec. 2.8. At first, the exterior derivative introduced in sec. A.13 canonically extends to the **V-valued exterior derivative**

$$\mathrm{d} : \Omega^k_P(V) \longrightarrow \Omega^{k+1}_P(V).$$

Since $\mathrm{d}^2 = 0$ still holds, a **V-valued de Rham complex** can be introduced just like in sec. A.13. Due to the additional structure of the vector space V, the V-valued exterior derivative can be refined. While G-equivariant forms comprise a subcomplex, the horizontal forms do not, since $\mathrm{d}\alpha$ by no means needs to be horizontal again, even if α is. The projection onto horizontal forms defines the **exterior covariant derivative**

$$\begin{aligned} \mathrm{d}^\# : \Omega^k_P(V)^\# &\longrightarrow \Omega^{k+1}_P(V)^\# \\ \alpha &\mapsto h^* \mathrm{d}\alpha \end{aligned},$$

thus the curvature form can be written as $\Omega = \mathrm{d}^\# \omega$, and the Bianchi identity reduces to $\mathrm{d}^\# \Omega = 0$, i.e. the curvature form is covariant constant. [KN63, §II.5] provides the necessary mathematical rigor. Unfortunately, $(\mathrm{d}^\#)^2 \neq 0$ in general, so there is no "covariant de Rham complex" as one might speculate—albeit a similar notion is reintroduced in the mathematical literature in a more general sense.

[g]It might seem awkward to denote both the curvature form Ω and the space of differential forms Ω^k_M with the same symbol. Unfortunately, tradition has a firm hold on mathematical notation, just like in physics.

The exterior covariant derivative can be restated more explicitly for basic forms. Let $\alpha \in \Omega^k_M(P \times_G V)$ be a k-form with values in the vector bundle $P \times_G V \xrightarrow{\pi'} M$ and $\tilde{\alpha} \in \Omega^k_P(V)^\flat$ the representing basic form on P under the isomorphism (2.1). Then one can show

$$\mathrm{d}^\# \tilde{\alpha} = \mathrm{d}\tilde{\alpha} + \rho(\omega) \wedge \tilde{\alpha} \in \Omega^{k+1}_P(V)^\flat, \tag{2.2}$$

where the "wedge product" $\wedge$ denotes both the wedge product of forms and the composition of the components of $\rho(\omega)$ with $\tilde{\alpha}$. It follows $(\mathrm{d}^\#)^2\tilde{\alpha} = \rho(\Omega) \wedge \tilde{\alpha}$ by further computation, thus the square of the exterior covariant derivative is proportional to the curvature. In other words, curvature can also be understood as a measure of how horizontal forms fail to form a complex.

2.6. Gauge fields

In order to make contact with the notions used in physics, consider local pull-backs of the connection 1-form $\omega \in \Omega^1_P(\mathfrak{g})$ by the canonical sections s_α introduced in sec. 2.4, which define $\mathfrak{g}$-valued 1-forms

$$A_\alpha := s_\alpha^* \omega \in \Omega^1_{U_\alpha}(\mathfrak{g})$$

locally on U_α, called **(local) gauge fields**. Conversely, one can prove, that the restriction ω_α of the connection 1-form ω to $P|_{U_\alpha}$ can be expressed as

$$\omega_\alpha = \mathrm{ad}_{g_\alpha^{-1}} \big(\pi^* A_\alpha\big) + g_\alpha^* \theta, \tag{2.3}$$

where θ is the Maurer-Cartan form introduced in (A.4) and $g_\alpha : P|_{U_\alpha} \longrightarrow G$ comes from the local trivialization $\phi_\alpha(p) = \big(x, g_\alpha(p)\big)$. The letter "$A$" is of course chosen in resemblance to the gauge fields encountered in electrodynamics, for example. Since physicists usually consider trivial bundles, e.g. a principal U(1)-product bundle over flat Minkowski space-time in classical electrodynamics, the process of trivializing the bundle is effectively hidden in the physical notation, since canonical global trivializations can be chosen in such situations.

More interesting topological properties of gauge fields are encoded in the global structure when considering nontrivial base spaces. On an overlap $U_{\alpha\beta}$ of local trivializations with transition function $g_{\alpha\beta}$ the respective local gauge fields are related by

$$\begin{aligned} A_\alpha &= \mathrm{ad}_{g_{\alpha\beta}} \big(A_\beta - g_{\alpha\beta}^* \theta\big) && \text{(in general)} \\ &= g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} - (\mathrm{d}g_{\alpha\beta}) g_{\alpha\beta}^{-1}. && \text{(for matrix groups)} \end{aligned} \tag{2.4}$$

If the local pull-backs A_α are glued together using this relation, this gives rise to a (global) form $A \in \Omega^1_M(\mathrm{ad}\, P)$. In the physical sense, A could be regarded as a global choice of gauge for the gauge bundle $P \xrightarrow{\pi} M$, whereas the mathematical literature just refers to A as the connection form, too.

Aside from the construction, this is already suggested when comparing the affine space of connections $\mathscr{A}$ with the space of gauges just mentioned. In other words, there is a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{connection 1-forms} \\ \omega \in \Omega^1_P(\mathfrak{g}) \end{array} \right\} \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \text{gauge choices} \\ A \in \Omega^1_M(\mathrm{ad}\, P) \end{array} \right\}$$

which is essentially due to the isomorphism (2.1). Therefore, a local Yang-Mills gauge theory in physics with gauge group G is essentially differential geometry of principal G-bundles, where the choice of a (physical) gauge is equivalent to the choice of a (mathematical) connection. Both the space of connections and the space of gauges on P will therefore be denoted as $\mathscr{A}$.

A **gauge transformation** of a principal G-bundle $P \xrightarrow{\pi} M$ is a G-equivariant diffeomorphism $\Phi : P \longrightarrow P$ such that the diagram

$$\begin{tikzcd} P \arrow[rr, "\Phi", "\approx"'] \arrow[dr, "\pi"'] & \circlearrowleft & P \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

commutes, where G-equivariance in this case means $\Phi(xg) = \Phi(x)g$ for all $g \in G$ and $x \in P$. In particular, such Φs map fibers to themselves and together with compositions constitute the **group of gauge transformations** $\mathscr{G}$. By considering a description of $\mathscr{G}$ in terms of a local trivialization, one finds $\mathscr{G} = \Gamma(\operatorname{Ad} P)$, i.e. the G-fiber bundle $\operatorname{Ad} P = P \times_{\operatorname{Ad}} G \xrightarrow{\tilde{\pi}} M$ associated to $P \xrightarrow{\pi} M$ via the conjugate mapping provides the space of gauge transformation.

Naturally, the question arises how a gauge transformation $\Phi : P \longrightarrow P$ acts on the space of connections $\mathscr{A}$ or rather how it affects the chosen gauge. From the mathematical point of view, this question is answered in [KN63, §II.6]. To give a satisfactory solution to this question, the effect on the local trivialization of the gauge bundle has to be investigated. Let $\phi_\alpha : P|_{U_\alpha} \longrightarrow U_\alpha \times G$ be a local trivialization with $g_\alpha : P|_{U_\alpha} \longrightarrow G$ being the mapping into the fiber part. Require the following diagram to commute:

$$\begin{tikzcd}
P|_{U_\alpha} \arrow[r, "\text{gauge transformation } \Phi", "\approx"'] \arrow[d, "\approx", "\substack{\text{local trivialization } \phi_\alpha \\ p \mapsto (x, g_\alpha(p))}"'] \arrow[dr, phantom, "\circlearrowleft"] & P|_{U_\alpha} \arrow[d, "\approx"', "\substack{\text{local trivialization } \phi_\alpha \\ p \mapsto (x, g_\alpha(\Phi(p)))}"] \\
U_\alpha \times G \arrow[r, "\substack{(x, g_\alpha(p)) \mapsto (x, \tilde{\Phi}_\alpha(p) g_\alpha(p)) \\ \text{formal effect of the gauge transformation}}"'] & U_\alpha \times G
\end{tikzcd}$$

This translates to $\tilde{\Phi}_\alpha(p) g_\alpha(p) = g_\alpha\big(\Phi(p)\big)$ as obvious from the diagram, and it follows that the introduced local "fiber gauge mapping" must take the explicit form

$$\begin{aligned} \tilde{\Phi}_\alpha : P|_{U_\alpha} &\longrightarrow G \\ p &\mapsto g_\alpha\big(\Phi(p)\big) g_\alpha^{-1}(p) \end{aligned}.$$

But due to G-equivariance of the composed mappings, it follows $\tilde{\Phi}_\alpha(pg) = \tilde{\Phi}_\alpha(p)$. Because of this G-invariance, i.e. the fact that $\tilde{\Phi}_\alpha$ does not depend on the fiber, there exists a mapping

$$\Phi_\alpha : U_\alpha \longrightarrow G,$$

such that $\tilde{\Phi}_\alpha(p) = \Phi_\alpha\big(\pi(p)\big) = \Phi_\alpha(x)$. The mapping Φ_α essentially describes the effect of the gauge transformation on the fiber with respect to the local trivialization ϕ_α.

If the connection is specified in the outlined geometrical manner by the choice of a distribution $H \subset \mathrm{T}P$, the Φ-gauge-transformed connection is defined in the obvious way as the push-forward $H^\Phi := \Phi_* H$. This translates to $\omega^\Phi = (\Phi^*)^{-1}\omega$ in the language of connection 1-forms. For the physicist the actual change of the local gauge field A_α is of more interest. Using equations (2.3) and (A.4), one can deduce the relation

$$\begin{aligned} A_\alpha^\Phi &= \operatorname{ad}_{\Phi_\alpha}\left(A_\alpha - \Phi_\alpha^* \theta\right) && \text{(in general)} \\ &= \Phi_\alpha A_\alpha \Phi_\alpha^{-1} - (\mathrm{d}\Phi_\alpha)\Phi_\alpha^{-1} && \text{(for matrix groups).} \end{aligned} \tag{2.5}$$

When comparing (2.4) and (2.5), it becomes obvious, that any gauge-invariant object constructed out of gauge fields will be well-defined globally on M.

2.7. Gauge field-strengths

Again, pulling back patches of the curvature 2-form $\Omega \in \Omega_P^2(\mathfrak{g})$ via the canonical sections $s_\alpha : U_\alpha \longrightarrow P|_{U_\alpha}$ yields the **gauge field-strengths**

$$F_\alpha := s_\alpha^* \Omega \in \Omega_{U_\alpha}^2(\mathfrak{g}).$$

Obviously, the structure equation in this case reads $F_\alpha = \mathrm{d}A_\alpha + \frac{1}{2}[A_\alpha, A_\alpha]$ and provides an easy way to calculate the field-strength from any given (local) gauge field. On overlaps $U_{\alpha\beta}$, the respective gauge field-strengths of the local gauge fields are related via

$$\begin{aligned} F_\alpha &= \operatorname{ad}_{g_{\alpha\beta}}(F_\beta) && \text{(in general)} \\ &= g_{\alpha\beta} F_\beta g_{\alpha\beta}^{-1}, && \text{(for matrix groups)} \end{aligned}$$

which makes their covariant nature manifest. If a gauge transformation $\Phi : P \longrightarrow P$ is performed on the principal G-bundle $P \xrightarrow{\pi} M$, the gauge field-strength changes according to

$$F_\alpha^\Phi = \mathrm{ad}_{\Phi_\alpha}(F_\alpha),$$

where $\Phi_\alpha : U_\alpha \longrightarrow G$ are the local functions of the gauge transformation introduced in the last section. It is important to remember, that a gauge transformation in the sense of sec. 2.6 in general is what physicists refer to as a "local gauge transformation". "Global gauge transformations" are only special cases of gauge transformations, where Φ acts in the same way on every fiber of $P \xrightarrow{\pi} M$.

Using the same line of reasoning as in sec. 2.6—analogous to the (local) gauge fields—the collection $\{F_\alpha\}$ defines a global **field-strength 2-form**

$$F \in \Omega^2_M(\mathrm{ad}\, P),$$

such that, in particular, there is again a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{curvature 2-forms} \\ \Omega \in \Omega^2_P(\mathfrak{g}) \end{array} \right\} \quad \stackrel{1:1}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \text{gauge field-strengths} \\ F \in \Omega^2_M(\mathrm{ad}\, P) \end{array} \right\}.$$

At this point it should become clear, that mathematicians and physicists essentially talk about the same things, but from very different viewpoints.[h]

2.8. Covariant derivative

The exterior covariant derivative introduced in sec. 2.5 can be pulled back under the isomorphism (2.1) to vector-bundle-valued forms on M. With respect to local trivializations, this yields the covariant derivative used in the physics literature.

This is not really a surprise: The exterior covariant derivative is well-defined on horizontal form and thus on basic forms, too. This defines the **covariant derivative on vector bundles**

$$\nabla : \Omega^k_M(P \times_\rho V) \longrightarrow \Omega^{k+1}_M(P \times_\rho V),$$

—where $\rho : G \longrightarrow \mathrm{GL}(V)$ is of course the linear representation of the gauge group—as the mapping equivalent to the covariant exterior derivative under the isomorphism (2.1). This can be shown as follows:

$$\begin{tikzcd} \cdots \arrow[r, "\mathrm{d}^\#"] & \Omega^k_P(V)^\flat \arrow[r, "\mathrm{d}^\#"] \arrow[d, equal, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^{k+1}_P(V)^\flat \arrow[r, "\mathrm{d}^\#"] \arrow[d, equal, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^{k+2}_P(V)^\flat \arrow[r, "\mathrm{d}^\#"] \arrow[d, equal, "\cong"'] & \cdots \\ \cdots \arrow[r, "\nabla"'] & \Omega^k_M(P \times_\rho V) \arrow[r, "\nabla"'] & \Omega^{k+1}_M(P \times_\rho V) \arrow[r, "\nabla"'] & \Omega^{k+2}_M(P \times_\rho V) \arrow[r, "\nabla"'] & \cdots \end{tikzcd}$$

Furthermore, it can be shown that ∇ is in fact a **skew-derivation**, which means for any $\alpha \in \Omega^k_M$ and $\beta \in \Omega^l_M(P \times_\rho V)$ the covariant derivative satisfies

$$\nabla(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^r \alpha \wedge \nabla\beta \in \Omega^{k+l}_M(P \times_\rho V),$$

which can be understood as a generalized version of the Leibniz product rule of differentiation.

To derive an explicit formula for the covariant derivative of a vector-bundle-valued k-form $\eta \in \Omega^k_M(P \times_\rho V)$, a collection $\{\eta_\alpha\}$ of local forms $\eta_\alpha \in \Omega^k_{U_\alpha}(V)$ is used again. As before, the local forms on any nonempty overlap $U_{\alpha\beta}$ are interrelated by

$$\eta_\alpha = \rho_{g_{\alpha\beta}}(\eta_\beta),$$

[h] In fact, these innate relations brought great benefit for both sides: Yang-Mills gauge theory became one of the most solid building blocks of theoretical physics. Essentially, the complete understanding of the microscopic world is based on local gauge field theory. Back in the '80s, gauge theory was a hot topic in geometry, too. Donaldson's famous theorem, as well as many other contributions to pure mathematics from this time period are more or less results from gauge theory. Even the modern methods in terms of Seiberg-Witten theory rely on gauge theory in the context of supersymmetry. [DK90] provides a nice introduction to many results of this era, with [Sco05] as a modern update in some parts, e.g. Seiberg-Witten theory.

and there exists a $\tilde{\eta} \in \Omega^k_P(V)^\flat$ such that $\eta_\alpha = s^*_\alpha \tilde{\eta}$. Working out the pull-back of $\mathrm{d}^\# \tilde{\eta}$ via s_α yields

$$\nabla \eta_\alpha = s^*_\alpha \mathrm{d}^\# \tilde{\eta} = \mathrm{d}\eta_\alpha + \rho(A_\alpha) \wedge \eta_\alpha.$$

Using the transformation properties of η_α and (local) gauge fields (2.4) on nonempty overlaps, it follows $\nabla \eta_\alpha = \rho_{g_{\alpha\beta}}(\nabla \eta_\beta)$, which justifies the name "covariant derivative" as used in the physics literature.

This construction of the covariant derivative also includes the covariant derivative of ordinary vector bundles. Given a real vector bundle $E \xrightarrow{\pi} M$, let $\mathrm{Fr}\, E \xrightarrow{\tilde{\pi}} M$ be the equivalent frame bundle, i.e. the corresponding principal $\mathrm{GL}_k(\mathbb{R})$-bundle. Using the standard representation Id of $\mathrm{GL}_k(\mathbb{R})$ on the real vector space fibers, the original bundle is recovered through the associate bundle $E = \mathrm{Fr}\, E \times_{\mathrm{Id}} V$. This immediately yields

$$\nabla : \Omega^k_M(E) \longrightarrow \Omega^{k+1}_M(E)$$

and in particular the special case $\nabla : \Gamma(E) \longrightarrow \Gamma(\mathrm{T}^* M \otimes E)$. Given a vector field $V \in \mathfrak{X}(M)$, this immediately yields the directional covariant derivative $\nabla_V : \Gamma(E) \longrightarrow \Gamma(E)$. In chap. 5 the exposition will resume at this point.

2.9. Example: Classical electrodynamics

Consider the abelian gauge group $G = \mathrm{U}(1)$ for an illustrative example: with respect to local coordinates $(x^1, \ldots, x^n)$ on U_α any gauge field can be expressed as $A_\alpha = \sum_i A_i \otimes \mathrm{d}x^i$, where $A_i : U_\alpha \longrightarrow \mathfrak{g}$ are the component functions. Note that α and i are two different sets of indices, that are not related—the α is just neglected in the components A_i. Since $\mathfrak{g} = \mathfrak{u}(1) = \mathrm{i}\mathbb{R} \cong \mathbb{R}$ is the Lie algebra in the specific case of interest, the local gauge fields A_α are in fact ordinary real 1-forms due to $\Omega^1_{U_\alpha}(\mathbb{R}) = \Omega^1_{U_\alpha}$, i.e.

$$\begin{aligned} A_\alpha &= \sum_i A_i \otimes \mathrm{d}x^i && (\text{for general } A^i : U_\alpha \longrightarrow \mathfrak{g}) \\ &= \sum_i A_i \, \mathrm{d}x^i && (\text{since } A_i : U_\alpha \longrightarrow \mathbb{R} \text{ for } G = \mathrm{U}(1)) \end{aligned}$$

The gauge field-strength then can be worked out explicitly with the exterior derivative introduced in sec. A.13,

$$\begin{aligned} F_\alpha &= \mathrm{d}A_\alpha + \frac{1}{2}[A_\alpha, A_\alpha] && (\text{in general}) \\ &= \sum_i \mathrm{d}(A_i \, \mathrm{d}x^i) + \frac{1}{2} \sum_{i,j} \left[A_i \, \mathrm{d}x^i, A_j \, \mathrm{d}x^j\right] && (\text{for } G = \mathrm{U}(1)) \\ &= \sum_{i,j} \frac{\partial A_i}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^i + \frac{1}{2} \sum_{i,j} A_i A_j \, \mathrm{d}x^i \wedge \mathrm{d}x^j \\ &= \frac{1}{2} \sum_{i,j} \left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \right) \mathrm{d}x^j \wedge \mathrm{d}x^i \\ &= \frac{1}{2} \sum_{i,j} (\partial_i A_j - \partial_j A_i) \, \mathrm{d}x^i \wedge \mathrm{d}x^j \\ &= \frac{1}{2} \sum_{i,j} F_{ij} \, \mathrm{d}x^i \wedge \mathrm{d}x^j, \end{aligned}$$

which of course yields the well-known antisymmetric field-strength tensor $F_{ij} = \partial_i A_j - \partial_j A_i$ from classical electrodynamics. The formalism developed up to this point essentially generalizes this concept to arbitrary gauge groups. For example, quantum chromodynamics just utilizes the different gauge group $G = \mathrm{SU}(3)$ instead of $\mathrm{U}(1)$, which makes the description much more complicated due to the non-vanishing Lie brackets.

CHAPTER 3

Cohomology and Characteristic Classes

Characteristic classes have become one of the most important tools in differential and algebraic geometry as well as algebraic topology. To each bundle over a manifold they assign certain cohomology classes, which essentially measure the extent to which the bundle is twisted—particularly, whether it possesses global sections or not. Thus, characteristic classes are global invariants that measure the deviation of the bundle from a global product structure under different aspects, which will be important when addressing certain existence questions in the succeeding chapters. However, a proper understanding requires some basic knowledge of algebraic topology, such that the first few sections are of a rather abstract and technical nature.

3.1. Abstract homology and cohomology

Let $C^\bullet := \{C^n\}_{n\in\mathbb{N}}$ be a sequence of abelian groups (or R-modules) connected by group (or module) homomorphisms $\delta^n : C^n \longrightarrow C^{n+1}$, called **coboundary operators**, such that the composition of any two consecutive maps vanishes, i.e. $\delta^{n+1} \circ \delta^n = 0$ for any $n \in \mathbb{N}$. Furthermore, let C^{-1} be the trivial group $\{e\}$. Sequences of this type are usually denoted as

$$0 \longrightarrow C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \cdots .$$

A sequence $C := (C^\bullet, \delta^\bullet)$ with $\delta^2 = 0$ starting at $C^{-1} := 0$ is called a **cochain complex**. There is an analogous construction called the **chain complex** $(C_\bullet, \partial_\bullet)$, where the homomorphisms map in the other direction and are called **boundary operators** $\partial_n : C_n \longrightarrow C_{n-1}$. In essence, everything is reversed as indicated in the chain sequence

$$0 \longleftarrow C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \cdots ,$$

albeit most texts usually introduce this latter definition first and develop the "co"-theory afterwards.[a] Details on this topic are featured in every textbook on algebraic topology, e.g. [Hat02, chp. 2].

Now, given a cochain complex $C = (C^\bullet, \delta^\bullet)$, one defines the subgroups (or submodules)

$$\textbf{\textit{n}-cocycles}\text{: } Z^n(C) := \ker(\delta^n : C^n \longrightarrow C^{n+1}) \subset C^n$$
$$\textbf{\textit{n}-coboundaries}\text{: } B^n(C) := \operatorname{im}(\delta^{n-1} : C^{n-1} \longrightarrow C^n) \subset C^n.$$

The n-boundaries $B_n(\tilde{C})$ and n-cycles $Z_n(\tilde{C})$ for chain complexes $\tilde{C} = (\tilde{C}_\bullet, \partial_\bullet)$ are defined analogously with respect to the boundary operator $\partial_\bullet$. The ***n*-th cohomology group (or module)** associated to the cochain complex C is then given as a factor group (or quotient module) of the form

$$H^n(C) := \frac{Z^n(C)}{B^n(C)} = \frac{\ker \delta^n}{\operatorname{im} \delta^{n-1}},$$

[a]Concerning the position of the indices, one might remember that of the homology boundary operators $\partial_n : C_n \longrightarrow C_{n-1}$, where the index is in lower position, actually decrease the grading index of the respective chain group (or module), whereas the upper position index of the coboundary operators $\delta^n : C^n \longrightarrow C^{n+1}$ indicates an increase of the cochain group (or module) index. This is a bit confusing regarding the contrary "co is below" rule used for the index notation in special relativity.

i.e. two n-cocycles are identified if they differ by a n-coboundary. The collection of all cohomology groups (or modules) $H^n(C)$ is called the **cohomology** of the chain complex C. The **homology** of a chain complex $\tilde{C}$ is defined analogously by factor groups (or quotient modules) of cocycles through boundaries:

$$H_n(\tilde{C}) = \frac{Z_n(\tilde{C})}{B_n(\tilde{C})} = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

There is a particularly important cochain complex constructed from a given chain complex as follows: Let $C = (C_\bullet, \partial_\bullet)$ be a chain complex of abelian groups (or modules) and let A be an abelian group (or module). Then the cochain complex $\operatorname{Hom}(C, A) := \big(\operatorname{Hom}(C, A)^\bullet, \delta^\bullet_{\operatorname{Hom}}\big)$ with coefficients in A is defined by

$$\operatorname{Hom}(C, A)^n := \operatorname{Hom}(C_n, A) = \big\{\text{group (or module) homomorphisms } C_n \longrightarrow A\big\},$$

where the coboundary mappings are defined by

$$\begin{aligned} \delta^n_{\operatorname{Hom}} : \operatorname{Hom}(C, A)^n &\longrightarrow \operatorname{Hom}(C, A)^{n+1} \\ f &\mapsto f \circ \partial_{n+1} \end{aligned}.$$

This essentially describes inserting the proper boundary operator ahead of the respective homomorphism as depicted in

$$\begin{tikzcd} \cdots & C_{n-1} \arrow[l, "\partial_{n-1}"'] & C_n \arrow[l, "\partial_n"'] \arrow[d, "\operatorname{Hom}(C,A)^n \ni f"'] \arrow[dr, phantom, "\circlearrowleft"] & C_{n+1} \arrow[l, "\partial_{n+1}"'] \arrow[dl, dashed, "f \circ \partial_{n+1} \in \operatorname{Hom}(C,A)^{n+1}"] & \cdots \arrow[l, "\partial_{n+2}"'] . \\ & & A & & \end{tikzcd}$$

Given a chain complex C, the **cohomology of C with A-coefficients** is defined to be the cohomology of the constructed cochain complex $\operatorname{Hom}(C, A)$, and denoted as

$$H^n(C; A) := H^n\big(\operatorname{Hom}(C, A)\big).$$

A **cochain map** $f^\sharp : C \longrightarrow \tilde{C}$ between cochain complexes C, $\tilde{C}$ is a collection of group (or module) homomorphisms $(f^\sharp)^n : C^n \longrightarrow \tilde{C}^n$ for $n \in \mathbb{N}$, such that each diagram

$$\begin{tikzcd} \cdots \arrow[r] & C^n \arrow[r, "\text{coboundary operator } \delta^n"] \arrow[d, squiggly, "\text{cochain homomorphism } (f^\sharp)^n"'] \arrow[dr, phantom, "\circlearrowleft"] & C^{n+1} \arrow[r] \arrow[d, squiggly, "\text{cochain homomorphism } (f^\sharp)^{n+1}"] & \cdots \\ \cdots \arrow[r] & \tilde{C}^n \arrow[r, "\text{coboundary operator } \tilde{\delta}^n"'] & \tilde{C}^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

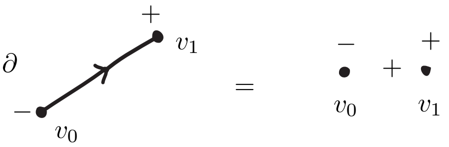
commutes, i.e. $(f^\sharp)^{n+1} \circ \delta^n = \tilde{\delta}^n \circ (f^\sharp)^n$ for all $n \in \mathbb{N}$. A cochain mapping induces an associated mapping f^* of the respective cohomologies, i.e. a collection of mappings

$$(f^*)^n : H^n(C) \longrightarrow H^n(\tilde{C})$$

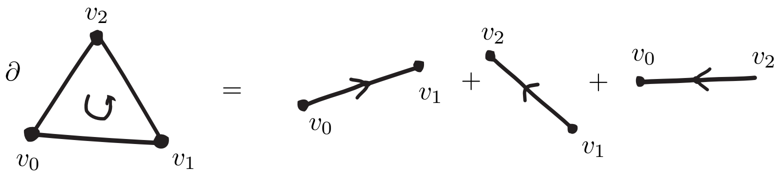
of the cohomology groups (or modules) for each $n \in \mathbb{N}$. Likewise, there is an analogous chain mapping for chain complexes that induces homology homomorphisms.

The great importance of cohomology stems from an associative, graded commutative product operation, effectively turning the cohomology of a cochain complex C into a graded ring, $H^\bullet(C)$, called the **cohomology ring** of C. The so-called **cup product** "$\smile$" is a method of adjoining two cocycles $\alpha \in Z^p(C)$ and $\beta \in Z^q(C)$ to form a composite cocycle $\alpha \smile \beta \in Z^{p+q}(C)$. After factorizing coborders, this construction extends to cohomology, where the cup product satisfies

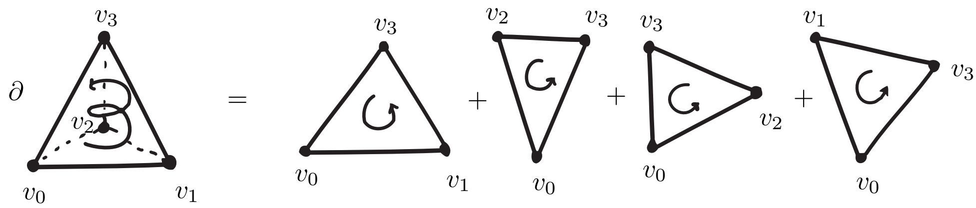
$$\alpha \smile \beta = (-1)^{pq}(\beta \smile \alpha)$$



$$\partial_1[v_0, v_1] = -[v_0] + [v_1]$$



$$\partial_2[v_0, v_1, v_2] = [v_0, v_1] + [v_1, v_2] + [v_2, v_0]$$



$$\partial_3[v_0, v_1, v_2, v_3] = [v_0, v_1, v_3] + [v_0, v_3, v_2] + [v_0, v_2, v_3] + [v_0, v_3, v_1]$$

FIGURE 3.1. Graphical representation of the boundary sums arising from 1-, 2- and 3-simplices.

for each $\alpha \in H^p(C)$ and $\beta \in H^q(C)$. The cup product is functorial, i.e. given a cochain mapping $f^\sharp : C \longrightarrow \tilde{C}$ with induced cohomology homomorphisms $(f^*)^n : H^n(C) \longrightarrow H^n(\tilde{C})$,

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$$

holds for all $\alpha \in H^p(C)$ and $\beta \in H^q(C)$. In other words, the induced cohomology mapping $f^* : H^\bullet(C) \longrightarrow H^\bullet(\tilde{C})$ is a (graded) ring homomorphism.

3.2. Singular and de Rham cohomology

The most important homology theory for topological spaces is constructed from simplices. A ***n*-simplex** $\triangle_n$ is the convex hull of a set of $n+1$ affinely independent points in some Euclidean space of dimension n or higher, which particularly inherits the topology induced from the Euclidean metric. Let X be a topological space, then consider continuous mappings $\sigma : \triangle_n \longrightarrow X$, which are called **singular *n*-simplices**. The boundary of such a singular n-simplex σ, denoted $\partial_n \sigma$, is defined to be the formal sum of the singular $(n-1)$-simplices represented by the restriction of σ to the faces of the standard n-simplex, with an alternating sign to take orientation into account, see fig. 3.1. In particular, the boundary of a 1-simplex σ is the formal difference $\sigma_1 - \sigma_0$ of the two (oriented) endpoints.

Let A be an abelian group (or module), then consider the free abelian groups generated by singular n-simplices, i.e.

$$C_n(X; A) := \sum_i A\sigma_i$$

where $\sigma_i : \triangle_n \longrightarrow X$ is a singular simplex. Together with the extension of the boundary operator ∂_n to such formal sums, the **singular chain complex** $C(X;A) = \big(C_\bullet(X;A), \partial_\bullet\big)$ is obtained. Via the general procedure described in the previous section, this yields the **singular homology** $H_n(X;A)$ with coefficients in A.

For R being any ring and the abelian group A being the integers $\mathbb{Z}$, applying the dualization procedure using the Hom-cochain complex construction yields a cochain complex $\mathrm{Hom}\big(C(X;\mathbb{Z}), R\big)$ with coboundar map δ_{Hom}. The corresponding **singular cohomology groups** $H^n(X;R)$ are R-modules, such that the **singular cohomology** $H^\bullet(X;R)$ can be given the structure of a graded algebra over R using the cup product. Since any ring R is an abelian group $A = (R,+)$ with respect to addition, the ring can be used directly for the definition of the chain complex C, i.e. one obtains both $H^\bullet(X;R)$ and $H_\bullet(X;R)$.

There is a deep relation between homology and cohomology, which in its most simple form can be expressed in terms of the Poincaré duality, cf. [Hat02, §3.3] or [Ful97, thm. 24.22]: For any compact oriented n-dimensional manifold M the k-th homology group is isomorphic to the $(n-k)$-th cohomology group for all $k \le n$, i.e.

$$H_k(M;R) \cong H^{n-k}(M;R).$$

This isomorphism is defined using the cap product "$\frown$", a method of adjoining p-chains α and q-chains β to form a $(p-q)$-form $\alpha \frown \beta$, which is sort of a inverse to the cup product, see [Hat02, §3.3]. The Poincaré duality can also be generalized to manifolds with boundary, where it is called the Poincaré-Lefschetz duality and there are even further extensions.

Now suppose $f : X \longrightarrow Y$ is a continuous mapping of topological spaces. This induces a mapping $f_\sharp$ of chain complexes via

$$\begin{tikzcd} \triangle_n \arrow[r, "\sigma\in C_n(X;R)"] \arrow[dr, dashed, "C_n(Y;R)\ni f\circ\sigma"'] & X \arrow[d, "f"] \arrow[dl, phantom, "\circlearrowleft"] \\ & Y \end{tikzcd} \quad\rightsquigarrow\quad \begin{array}{c} (f_\sharp)_n : C_n(X;R) \longrightarrow C_n(Y;R) \\ \sigma \mapsto f\circ\sigma \end{array},$$

which in turn induces a homology mapping $f_* : H_\bullet(X;R) \longrightarrow H_\bullet(Y;R)$. Dualizing to cochains, the respective cohomology mapping is $f^* : H^\bullet(Y;R) \longrightarrow H^\bullet(X;R)$. As before, the cohomology mapping is reverse to the homology mapping. In particular, if $f : X \overset{\approx}{\longrightarrow} Y$ is a homeomorphism of topological spaces (i.e. f and f^{-1} are continuous with respect to the topologies of either space) it induces isomorphisms of the associated singular (co-)homologies. This can be visualized as follows:

$$\begin{tikzcd} H_n(X;A) \arrow[d, "(f_*)_n", "\cong"'] & & & & H^n(X;A) \\ & \underset{\text{isomorphism}}{\overset{\text{induced homology}}{\Longleftarrow}} & X \arrow[r, "\text{homeomorphism } f", "\approx"'] & Y \quad \underset{\text{isomorphism}}{\overset{\text{induced cohomology}}{\Longrightarrow}} & \\ H_n(Y;A) & & & & H^n(Y;A) \arrow[uu, "(f^*)^n", "\cong"'] \end{tikzcd}$$

Thus, the singular homology and cohomology groups of a topological space are in fact **topological invariants**, i.e. invariant under homeomorphisms of the respective spaces.

Given a topological space X and a subset $Y \subset X$, the pair (X,Y) is called a **pair of spaces**. Let (X_1,Y_1) and (X_2,Y_2) be two pairs of spaces, then $f : X_1 \longrightarrow X_2$ is called a **space pair mapping** if $f(Y_1) \subseteq Y_2$. Such a mapping is usually denoted $f : (X_1,Y_1) \longrightarrow (X_2,Y_2)$ to indicate the preservation of the subsets Y_1 and Y_2. Given a pair of spaces (X,Y) one considers the factor group (or quotient module) chain complex $C_n(X,Y) := C_n(X;\mathbb{Z})/C_n(Y;\mathbb{Z})$ with $\mathbb{Z}$-coefficients. The associated cohomology theory with groups

$$H^n(X,Y;R) = H^n\Big(\mathrm{Hom}\big(C(X,Y);R\big)\Big)$$

is called the **relative (singular) cohomology** $H^\bullet(X,Y;R)$ of (X,Y) with R-coefficients. Note that this construction is specific for singular cohomology and in general cannot be applied to other cohomology theories.

Relative cohomology can be used to represent a choice of orientation via the choice of a generator of the top cohomology group: Let V be a vector space and $V_0 := V \setminus \{0\}$ be the same vector space with zero removed. Then the choice of orientation for V is equivalent to the choice of one of the two generators of the group $H^n(V,V_0;\mathbb{Z}) \cong \mathbb{Z}$—remember that $\mathbb{Z}$ is generated by either $+1$ or -1. Let $E \xrightarrow{\pi} M$ be a vector bundle and E_0 be the total space E with the zero section removed. Then one can choose a cohomology class $u \in H^n(E, E_0; \mathbb{Z})$ whose restriction via $H^n(E, E_0; \mathbb{Z}) \longrightarrow H^n(F, F_0; \mathbb{Z})$ to each fiber $F := E_p$ of the vector bundle, which is induced by the inclusion mapping $(F, F_0) \hookrightarrow (E, E_0)$ of each fiber, gives the vector space orientation of F. This defines the orientation of the bundle in terms of a certain cohomology class and should be compared to the more elementary definition given in sec. A.2. Of course, not every bundle is orientable and in such cases the highest cohomology group $H^n(E, E_0; \mathbb{Z})$ is just the trivial group.

Now suppose, that the topological space M is in fact a smooth manifold, such that differential forms are available. The most important consequence of the exterior differentiation, which is due to the antisymmetry of the wedge product, is $\mathrm{d}(\mathrm{d}\omega) = 0$ for any differential form ω. Any k-form ω satisfying $\mathrm{d}\omega = 0$ is called **closed**. If, in addition, ω can be written as $\omega = \mathrm{d}\sigma$ for a suitable $(k-1)$-form σ, then ω is called **exact**—that is, exact k-forms are closed and in fact exterior differentiations of $(k-1)$-forms. In particular, closed forms are elements of the kernel of the exterior differentiation, whereas exact forms are in the image.

Naturally, the question arises which essential different types of closed differential forms there are on a given manifold, if one ignores differences by exact forms. A more accessible interpretation of this question is provided in the next section. By construction, this information is encoded in the ***p*-th de Rham cohomology group** $H^p_{\mathrm{dR}}(M)$. On a real differentiable manifold M with exterior differentiation $\mathrm{d}_k : \Omega^k_M \longrightarrow \Omega^{k+1}_M$ these groups are defined as quotient $\mathbb{R}$-vector spaces

$$H^p_{\mathrm{dR}}(M) := \frac{\{\text{closed } p\text{-forms}\}}{\{\text{exact } p\text{-forms}\}} = \frac{\ker \mathrm{d}_p}{\operatorname{im} \mathrm{d}_{p-1}}.$$

The **de Rham cohomology** fits easily into the general scheme described in the last section: The differential forms on any smooth manifold M form $\mathbb{R}$-vector spaces $\Omega^k(M)$ for each $k \in \mathbb{N}$, and the exterior derivative $\mathrm{d}^k : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ provides coboundary mappings. Thus, the **de Rham cochain complex** is defined by $\big(\Omega^\bullet(M), \mathrm{d}^\bullet\big)$, which gives the de Rham cohomology groups $H^n_{\mathrm{dR}}(M)$ when applied to the general construction outlined before. Obviously, the cup product of the de Rham cohomology ring is given by the wedge product of differential forms.

In the case of certain manifolds the de Rham and singular cohomology are equivalent: de Rham's theorem, which was proved by de Rham back in 1931, states that for any compact oriented smooth manifold M, the de Rham cohomology groups $H^n_{\mathrm{dR}}(M)$ and the singular cohomology groups with real coefficients $H^n(M; \mathbb{R})$ are isomorphic as real vector spaces.[b]

3.3. Example: de Rham cohomology and classical vector analysis

To show the usefulness of the concept of cohomology and homology theory, a detailed example with direct application to classical electrodynamics is discussed. Considering a 3-dimensional spatial volume, a force field $\vec{F}(\vec{r})$ is called **conservative** if the mechanical work

$$W = \oint_\gamma \vec{F} \cdot \mathrm{d}\vec{s}$$

vanishes for any closed curve γ. This statement is obviously true from Stokes' theorem if the force field comes from a scalar potential, i.e. if there exists a real-valued smooth function Φ with $\vec{F}(\vec{r}) = \operatorname{grad} \Phi(\vec{r})$. Naturally, the question arises, whether such a potential Φ exists for a given force field.

Consider an open subset $U \subset \mathbb{R}^3$ with standard coordinates (x^1, x^2, x^3). As mentioned, the spaces $\Omega^k(U)$ of differential k-forms on U are real vector spaces. In particular, 0- and

[b] Many further useful cohomology theories are known, e.g. the Alexander-Spanier cohomology—essential a dual construction to the de Rham cohomology in terms of compact supports for the differential forms—or the Čech-cohomology, which is formulated via sheafs and coverings. For smooth manifolds both cohomology theories are isomorphic to the singular and de Rham cohomology, i.e. one has many possibilities for actual calculation of the cohomology groups.

3-forms can be identified with real-valued functions $C^\infty(U)$, whereas 1- and 2-forms can be understood as vector fields $\mathfrak{X}(U)$, i.e.

$$\begin{aligned}\Omega^0(U) &\cong \Omega^3(U) \cong C^\infty(U)\\ \Omega^1(U) &\cong \Omega^2(U) \cong C^\infty(U,\mathbb{R}^3) \cong \mathfrak{X}(U)\end{aligned}$$

as $\mathbb{R}$-vector spaces. Intuitively, this can be understood due to $\mathrm{T}_pU \cong \mathbb{R}^3$ for each $p \in U \subset \mathbb{R}^3$ and the dimensions

$$\begin{aligned}\dim_{\mathbb{R}} \Lambda^0\mathbb{R}^3 &= \dim_{\mathbb{R}} \Lambda^3\mathbb{R}^3 = 1\\ \dim_{\mathbb{R}} \Lambda^1\mathbb{R}^3 &= \dim_{\mathbb{R}} \Lambda^2\mathbb{R}^3 = 3\end{aligned}$$

of the exterior algebra. The respective bases are then either $(\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3)$ in the case of $\Omega^1(U)$ or $(\mathrm{d}x^2 \wedge \mathrm{d}x^3, \mathrm{d}x^3 \wedge \mathrm{d}x^1, \mathrm{d}x^1 \wedge \mathrm{d}x^2)$ for $\Omega^2(U)$. The explicit isomorphisms are provided in [JK06, §10.2].

For a smooth function $f \in C^\infty(U) \cong \Omega^0(U)$ the exterior derivative reads

$$\mathrm{d}f = \frac{\partial f}{\partial x^1}\,\mathrm{d}x^1 + \frac{\partial f}{\partial x^2}\,\mathrm{d}x^2 + \frac{\partial f}{\partial x^3}\,\mathrm{d}x^3,$$

which is obviously the **gradient** of f with respect to basis vectors $\mathrm{d}x^1$, $\mathrm{d}x^2$, $\mathrm{d}x^3$. Similarly, for a 1-form $\omega \in \Omega^1(U) \cong C^\infty(U,\mathbb{R}^3)$ with $\omega = \omega^1\,\mathrm{d}x^1 + \omega^2\,\mathrm{d}x^2 + \omega^3\,\mathrm{d}x^3$ one has

$$\mathrm{d}\omega = \left(\frac{\partial \omega^3}{\partial x^2} - \frac{\partial \omega^2}{\partial x^3}\right)\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \left(\frac{\partial \omega^1}{\partial x^3} - \frac{\partial \omega^3}{\partial x^1}\right)\mathrm{d}x^3 \wedge \mathrm{d}x^1 + \left(\frac{\partial \omega^2}{\partial x^1} - \frac{\partial \omega^1}{\partial x^2}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2,$$

i.e. the **rotation** of the 1-form ω when regarded as a vector field. In three dimensions a vector field on U can also be formulated as a 2-form $\eta \in \Omega^2(U) \cong C^\infty(U,\mathbb{R}^3)$ with component representation $\eta = \eta^{23}\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \eta^{31}\,\mathrm{d}x^3 \wedge \mathrm{d}x^1 + \eta^{12}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$, whose exterior derivative is the **divergence** of η as seen from

$$\mathrm{d}\eta = \left(\frac{\partial \eta^{23}}{\partial x^1} + \frac{\partial \eta^{31}}{\partial x^2} + \frac{\partial \eta^{12}}{\partial x^3}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3.$$

This equivalence of the exterior derivative to the operators grad, rot and div from classical vector analysis can be shown in a diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0(U) \arrow[r, hook, "\mathrm{d}"] \arrow[d, equal, "\cong"'] & \Omega^1(U) \arrow[r, "\mathrm{d}"] \arrow[d, equal, "\cong"'] & \Omega^2(U) \arrow[r, two heads, "\mathrm{d}"] \arrow[d, equal, "\cong"'] & \Omega^3(U) \arrow[r] \arrow[d, equal, "\cong"'] & 0\\
& C^\infty(U) \arrow[r, hook, "\mathrm{grad}"'] \arrow[rr, dashed, bend right, "\mathrm{rot}\,\mathrm{grad}\,f=0"'] & C^\infty(U,\mathbb{R}^3) \arrow[r, "\mathrm{rot}"'] \arrow[rr, dashed, bend right, "\mathrm{div}\,\mathrm{rot}\,\vec{a}=0"'] & C^\infty(U,\mathbb{R}^3) \arrow[r, two heads, "\mathrm{div}"'] & C^\infty(U) &
\end{tikzcd}$$

The exactness of the de Rham chain complex (i.e. $\mathrm{d}_n^2 = 0$ for all n) immediately yields the well-known identities $\mathrm{rot}\,\mathrm{grad}\,f = 0$ and $\mathrm{div}\,\mathrm{rot}\,\vec{a} = 0$ found in most textbooks.

Since the exterior differentiation is well-defined for any smooth manifold, even without reference to any local coordinates, it provides a natural generalization (or rather proper mathematical definition) of the classical vector analysis operators grad, rot and div. Let M be a smooth manifold, then rearranging the above diagram in the general case $U = M$ to

$$\begin{tikzcd}
\Omega^0(M) \arrow[r, hook, "\mathrm{d}^0"] \arrow[d, equal, "\cong"'] & \Omega^1(M) \arrow[d, equal, "\cong"'] \arrow[d, bend left=60, "\mathrm{d}^1"]\\
\Omega^3(M) & \Omega^2(M) \arrow[l, two heads, "\mathrm{d}^2"]
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
C^\infty(M) \arrow[r, hook, bend left, "\mathrm{grad}"] & C^\infty(M,\mathbb{R}^3) \arrow[l, two heads, bend left, "\mathrm{div}"] \arrow[loop right, "\mathrm{rot}"]
\end{tikzcd}$$

makes the identification between the exterior differentiation and the operators of vector analysis even more accessible.

Thus, the question whether a potential exists for a given force field (i.e. vector field) should be reformulated in terms of de Rham cohomology. Using the above identification, there is a direct interpretation of the de Rham cohomology groups:

$$H^1_{\mathrm{dR}}(M) = \frac{\ker \mathrm{d}^1}{\operatorname{im} \mathrm{d}^0} = \frac{\{\text{rotation-free vector fields}\}}{\{\text{gradient vector fields}\}}$$
$$H^2_{\mathrm{dR}}(M) = \frac{\ker \mathrm{d}^2}{\operatorname{im} \mathrm{d}^1} = \frac{\{\text{divergence-free vector fields}\}}{\{\text{rotation vector fields}\}}$$

For these groups to vanish (i.e. to be the trivial group), both the "denominator" and "numerator" have to be equal. Thus, if $H^1_{\mathrm{dR}}(M) = 0$, every rotation-free vector field on M can be written as the gradient of some smooth function $f \in C^\infty(M)$. If $H^2_{\mathrm{dR}}(M) = 0$ any divergence-free vector field can be understood as being a pure rotational field.[c]

All the de Rham cohomology groups of $\mathbb{R}^n$ are known as a result of the Poincaré lemma, which states that for all $n \in \mathbb{N}$ the groups are

$$H^n_{\mathrm{dR}}(\mathbb{R}^k) \cong \begin{cases} \mathbb{R} & : \quad k = 0 \\ 0 & : \quad \text{else} \end{cases} .$$

A proof is found in [BT82, chp. 4]. For the physically particularly interesting case of $\mathbb{R}^3$ this implies that every globally defined rotation-free vector field is actually a gradient field and any divergence-free vector field is rotation-induced.

Note that the presence of an electrical point charge yields a singularity due to the $\frac{1}{r^2}$-behavior of the field strength, thus such a vector field is not defined on $\mathbb{R}^3$ but rather the "punctured volume" $\mathbb{R}^3 \setminus \{p\}$ with the singular point removed. One can calculate

$$H^n_{\mathrm{dR}}\big(\mathbb{R}^k \setminus \{0\}\big) \cong H^n_{\mathrm{dR}}(S^{k-1}) \cong \begin{cases} \mathbb{R} & : \quad n = 0 \text{ or } k-1 \\ 0 & : \quad \text{else} \end{cases}$$

which implies $H^1_{\mathrm{dR}}(\mathbb{R}^3 \setminus \{p\}) \cong 0$ and $H^2_{\mathrm{dR}}(\mathbb{R}^3 \setminus \{p\}) \cong \mathbb{R}$. Thus, any rotation-free electrical field strength always comes from a gradient potential, but not all divergence-free (magnetic) vector fields are purely rotational.

It is interesting to note, how the assumption of point charges already gives problems even at this elementary level. Thus, string theory's fundamental concept to remedy any point-like elementary structures is beneficial even for classical electrodynamics and realized via smooth charge distributions ρ.

3.4. Betti and Hodge numbers

The rank of the singular cohomology groups is denoted $b^p := \operatorname{rank} H^p(M;\mathbb{R})$, which are called the **Betti numbers**. Using the isomorphism $H^p(M;\mathbb{R}) \cong H^p_{\mathrm{dR}}(M)$ as provided by de Rham's theorem in the case of a compact oriented smooth manifold M, this definition reduces to the dimensions

$$b_p := \dim_{\mathbb{R}} H^p_{\mathrm{dR}}(M)$$

of real vector spaces for the de Rham cohomology. The Betti numbers are important topological invariants of a differentiable manifold (or any topological space in the more general sense) with many geometric properties:

- Obviously, $b_k = 0$ on a n-dimensional manifold for any $k > n$. This follows from the fact, that there are no k-forms on a n-dimensional manifold for $k > n$.
- The zeroth Betti number b_0 gives the number of connection components. In particular, $b_0 = 1$ for any (path-)connected topological space.

[c]It should be pointed out, that the non-vanishing of these cohomology groups does not imply the non-existence of the respective scalar or vector potentials—their existence is just not guaranteed as in the case of vanishing de Rham cohomology groups. Thus, even in very non-trivial spatial volumes M with non-trivial cohomology groups $H^1_{\mathrm{dR}}(M)$ and $H^2_{\mathrm{dR}}(M)$ there might exist conservative or pure rotational forces.

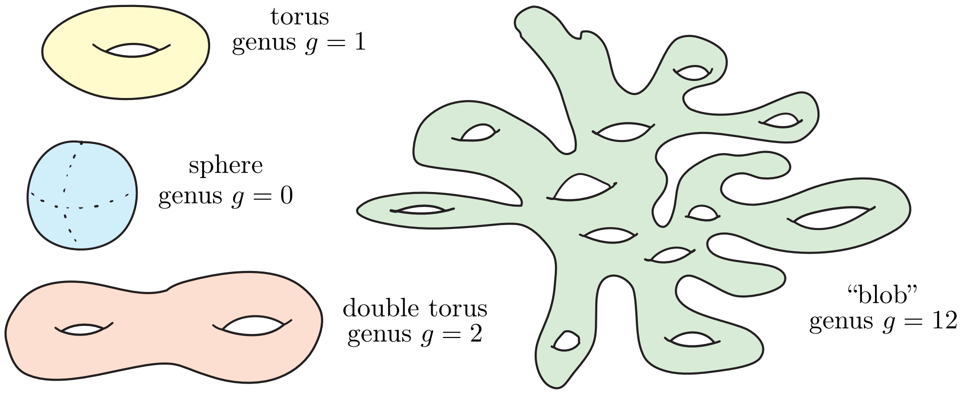


FIGURE 3.2. Oriented Riemann surfaces with specified genus.

- The first Betti number b_1 is the rank of the fundamental group. It can be interpreted as the number of distinct non-contractible loops on the manifold.
- The k-th Betti number encodes the number of essentially different closed k-forms, i.e. equivalence up to linearity and differences by exact forms. The physical interpretation of this topological information in the case of 3-manifolds was discussed in the last section, i.e. given a 3-manifold M, the vanishing $b_1(M) = b_2(M) = 0$ implies, that divergence-free vector fields are purely rotational, whereas rotation-free vector fields are given by gradients of scalar potentials.
- For any orientable, closed n-dimensional manifold the symmetry $b_k = b_{n-k}$ holds, thus the middle Betti number is often the most important. This is closely related to the Poincaré duality mentioned in sec. 3.1. Note that this implies $b_1 = b_2$ in the case of suitable 3-manifolds, which is important for the implications discussed in the last section of the vanishing of either the first or second de Rham cohomology group.
- For any orientable, closed 2-dimensional surface (e.g. Riemann surface) the Betti numbers are $b_0 = 1$, $b_1 = 2g$, $b_2 = 1$, where g is the genus of the surface. The **genus** is originally defined via the dimension of certain sheaf cohomology group, see [For81, chp. 2] for the detailed construction. Intuitively, the genus is the number of holes in a closed surface, e.g. the sphere has $g = 0$, the torus $g = 1$, etc., see fig. 3.2.
- The alternating sum of all Betti numbers gives the Euler characteristic of the manifold, which describes the minimal number of zeros any global vector field on the manifold has. This will be reviewed later in the context of characteristic classes.

For complex manifolds instead of the exterior differentiation d the antiholomorphic[d] exterior differentiation operator $\bar{\partial}_{r,s} : \Omega_N^{r,s} \longrightarrow \Omega_N^{r,s+1}$ is used to define the notions of **$\bar{\partial}$-exact** and **$\bar{\partial}$-closed** forms. Again, $\bar{\partial}(\bar{\partial}\eta) = 0$, thus it makes sense to define the **(r, s)-th Dolbeault cohomology group** via

$$H_{\bar{\partial}}^{r,s}(M) := \frac{\{\bar{\partial}\text{-closed } (r,s)\text{-forms}\}}{\{\bar{\partial}\text{-exact } (r,s)\text{-forms}\}} = \frac{\ker \bar{\partial}_{r,s}}{\operatorname{im} \bar{\partial}_{r,s-1}}.$$

The corresponding dimensions $h^{r,s} := \dim_{\mathbb{C}} H_{\bar{\partial}}^{r,s}(M)$ are called the **Hodge numbers** of M and are—analogous to the Betti numbers—important topological invariants of the complex manifold. The Hodge numbers are refinements of the Betti numbers, as $b_k = \sum_{r+s=k} h^{r,s}$ holds.

---

[d] In principle, this procedures would go through with the holomorphic differentiation operator ∂ as well. But, it is important to realize, that the $\bar{\partial}$-operator is a natural extension of the notion of holomorphic functions (which can be stated as $\bar{\partial} f = 0$) to differential forms. Thus, the Dolbeault cohomology of the $\bar{\partial}$-operator has many further implication for the study of holomorphic forms, which is beyond the scope of this text. For further investigations the reader is referred to the introductory text [Huy05].

The real value of the cohomological formulation comes from the numerous algebraic properties typical for all cohomology theories. In very general circumstances most of the known cohomology theories are in fact equivalent (or at least related), thus one can choose a specific cohomology theory (e.g. singular cohomology, Dolbeault cohomology, de Rham cohomology, etc.) which is particularly suited to calculate the cohomology groups under the given circumstances, but interpret the results by the de Rham cohomology groups, for example. A very readable general introduction to cohomology theory is found in [Hat02, chp. 3].

3.5. Stiefel-Whitney classes

The introduction to algebraic topology ends at this point, and attention is turned to actual applications of the general theory, which are utilized in the following chapters. Characteristic classes have become an important tool in topology and geometry and are formulated in terms of cohomology classes as developed in the last sections. Essentially, they measure the twisting and non-triviality of certain vector bundles under different aspects and come in four flavors:

for real vector bundles $E \xrightarrow{\pi} M$:	Stiefel-Whitney classes $w_i(E) \in H^i(M;\mathbb{Z}_2)$
for complex vector bundles $E \xrightarrow{\pi} M$:	Chern classes $c_i(E) \in H^{2i}(M;\mathbb{Z})$
for real vector bundles $E \xrightarrow{\pi} M$:	Pontryagin classes $p_i(E) \in H^{4i}(M;\mathbb{Z})$
for an oriented real n-vec. $E \xrightarrow{\pi} M$:	Euler class $e(E) \in H^n(M;\mathbb{Z})$

In this section the first set of characteristic classes, the Stiefel-Whitney classes will be introduced. There are two main approaches to the theory of characteristic classes: the differential geometric along the lines of Chern-Weil theory, and the algebraic approach. Following the expositions in either [MS74, §4] or [Hat03, §3.1], the characteristic classes are defined algebraically via a set of four axioms:

(1) For every real vector bundle $E \xrightarrow{\pi} M$ there corresponds a sequence of cohomology classes
$$w_i(E) \in H^i(M;\mathbb{Z}_2)$$
with $\mathbb{Z}_2$-coefficients, called the **Stiefel-Whitney classes** of E. A construction exists which guarantees existence. The zeroth class $w_0(E)$ is equal to the unit element of $H^0(M;\mathbb{Z}_2)$, and $w_i(E) = 0$ for $i > n$ if $E \xrightarrow{\pi} M$ is a n-vector bundle.

(2) Let $E \xrightarrow{\pi} M$, be a vector bundle and $g : N \longrightarrow M$ a continuous mapping. If g^*E is the pullback vector bundle over N, then
$$w_i(g^*E) = g^* w_i(E).$$

(3) Let $E \xrightarrow{\pi} M$, $E' \xrightarrow{\pi'} M$ be two vector bundles over the same base space, then there is the **Whitney product formula**
$$w_k(E \oplus E') = \sum_{i=0}^{k} w_i(E) \smile w_{k-i}(E').$$

(4) The tautological line bundle $\gamma^1 \xrightarrow{\mathrm{pr}_1} \mathbb{R}\mathcal{P}^1$ (Möbius strip) has non-vanishing first Stiefel-Whitney class, i.e. $w_1(\gamma^1) \in H^1(\mathbb{R}\mathcal{P}^1;\mathbb{Z}_2) \cong \mathbb{Z}_2$ is the generator of the group.

It requires considerable work to show that these axioms in fact uniquely define the Stiefel-Whitney classes, cf. [MS74, §8] or [Hat03, thm. 3.1]. Given a manifold M, the characteristic classes are defined with respect to the manifold's tangent bundle $\mathrm{T}M \xrightarrow{\pi} M$, i.e.
$$w_i(M) := w_i(\mathrm{T}M) \in H^i(M;\mathbb{Z}_2).$$

Furthermore, one defines the **total Stiefel-Whitney class** in the cohomology ring $H^\bullet(M;\mathbb{Z}_2)$ of the base space with $\mathbb{Z}_2$-coefficients as
$$w(E) := \sum_{i=0}^{\infty} w_i(E) = 1 + w_1(E) + \ldots + w_n(E) + 0 + \ldots$$

The Whitney product formula then reduces to $w(E \oplus F) = w(E) \smile w(F)$, where "$\smile$" is the canonical extension of the cup product to the cohomology ring.

Alternatively, the Stiefel-Whitney classes can be understood as certain coefficients of an algebraic equation via the Leray-Hirsch theorem, see [BT82]. As a highly nontrivial result, it states that the cohomology ring of the associated frame bundle $H^\bullet\big(\mathrm{Fr}(E); \mathbb{Z}_2\big)$ is in fact a free $H^\bullet(M; \mathbb{Z}_2)$-module with basis $(1, x, \ldots, x^{n-1})$, where x^k is to be understood as the k-th power of x. Thus, x^n can be uniquely represented as a linear combination of the basis elements, which leads to

$$x^n + w_1(E)x^{n-1} + \ldots + w_n(E)1 = 0.$$

The geometric data encoded in the Stiefel-Whitney classes is of enormous value and is summarized in the following list:

- If $\varepsilon := M \times \mathbb{R}^n \xrightarrow{\pi} M$ is a trivial product vector bundle on M, the Stiefel-Whitney classes are $w_i(\varepsilon) = 0$ for all $i > 0$. Using the Whitney product formula it immediately follows $w_i(\varepsilon \oplus E) = w_i(E)$ for all i, i.e. the Stiefel-Whitney classes are oblivious to trivial bundles. Conversely, since adding product bundles does not change the value of the Stiefel-Whitney classes, this is sometimes referred to as the "stability of the Stiefel-Whitney classes".
- Let $E \xrightarrow{\pi} M$ be a real n-vector bundle which possesses k linearly independent nowhere zero global sections, then the upper k Stiefel-Whitney classes vanish, i.e.
$$w_{n-k+1}(E) = w_{n-k+2}(E) = \ldots = w_n(E) = 0.$$
Thus, the Stiefel-Whitney classes measure "how much of the bundle is trivial".
- All the Stiefel-Whitney classes $w_i(M) = w_i(\mathrm{T}M)$ of a compact smooth manifold M vanish if and only if the manifold is a boundary of a compact smooth manifold.

However, the properties that will become most important in the spinor geometry chapter are the following:

- The first Stiefel-Whitney class $w_1(E)$ is zero if and only if the bundle is orientable. In particular, a smooth manifold M is orientable if and only if $w_1(M) = w_1(\mathrm{T}M) = 0$.
- The second Stiefel-Whitney class $w_2(E)$ is zero if and only if the bundle admits a spin structure.
- The third Stiefel-Whitney class $w_3(E)$ is zero if and only if the bundle admits a spin$^{\mathbb{C}}$ structure.

Spin and spin$^{\mathbb{C}}$ structures will be introduced in chap. 4. For the moment, it is enough to remark that the Stiefel-Whitney classes provide necessary existence information.

3.6. Euler characteristic and Euler class

Another important topological invariant is the Euler characteristic, which was originally introduced for polyhedras in the 18th century. Euler found the relation $\chi = V - E + F = 2$ for all polyhedra homeomorphic to a 2-sphere, where V, E, and F are respectively the numbers of vertices (corners), edges and faces. Using the formalism of simplicial and singular (co-)homology, the Euler characteristic can be generalized to any topological space by defining

$$\chi(M) := \sum_{i=0}^{\infty} (-1)^i b^i(M)$$

if this sum exists, i.e. the **Euler characteristic** is the alternating sum of the Betti numbers. In particular, for a compact oriented smooth n-manifold, due to $H^n(M; \mathbb{R}) \cong H^n_{\mathrm{dR}}(M)$, the Euler characteristic

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{R}} H^i_{\mathrm{dR}}(M)$$

equals the alternating sum of the dimensions of the de Rham cohomology groups.

The Euler characteristic is one of the most powerful topological invariants[e] of a manifold due to its simple definition. Given two topological spaces X and Y, there are the relations

$$\begin{aligned} &\text{union:} && \chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y) \\ &\text{cartesian product:} && \chi(X \times Y) = \chi(X)\chi(Y) \end{aligned}$$

From the Poincaré duality it follows $\chi(M) = 0$ for any closed odd-dimensional manifold M. The Euler characteristic of a closed orientable surface M can be calculated from its genus g as

$$\chi(M) = 2 - 2g(M).$$

As the string world-volumes in the later chapters are closed 2-dimensional manifolds (in fact Riemann surfaces, i.e. connected 1-dimension complex manifolds), this identity becomes quite important in the context of string interactions.

The extension of the Euler characteristic to bundles—understood as fibrations of the total space E—is called the Euler class. Suppose the real n-vector bundle $E \xrightarrow{\pi} M$ is oriented, then there exists a certain refinement of the top Stiefel-Whitney class $w_n(E)$ in the cohomology with $\mathbb{Z}$-coefficients, which encapsulates the additional information of orientation. The actual construction of the Euler class requires quite advanced concepts of algebraic topology, which can only be outlined here, see [MS74, §§9-10] or [Hat03, §3.2] for details.

As described in sec. 3.2, the choice of orientation for $E \xrightarrow{\pi} M$ corresponds to a relative cohomology class $u \in H^n(E, E_0; \mathbb{Z})$. The inclusion mapping $\iota : (E, \emptyset) \hookrightarrow (E, E_0)$ gives rise to the cohomology mapping

$$\iota^* : H^n(E, E_0; \mathbb{Z}) \longrightarrow H^n(E; \mathbb{Z}).$$

Due to an isomorphism $j : H^n(E; \mathbb{Z}) \xrightarrow{\cong} H^n(M; \mathbb{Z})$, the orientation class u defines a cohomology class in $e(E) := j \circ \iota^* u \in H^n(M; \mathbb{Z})$, called the **Euler class** of $E \xrightarrow{\pi} M$. The basic properties of the Euler class can be summarized as follows:

- Reversing the orientation of the vector bundle $E \xrightarrow{\pi} M$ changes the sign of $e(E)$. For oriented n-vector bundles with n odd it follows $e(E) + e(E) = 0 \in H^n(M; \mathbb{Z})$, as every odd-dimensional vector bundles possesses a orientation-reversing automorphism, thus $e(E) = -e(E) \iff e(E) + e(E) = 0$. Therefore, the Euler class is sensitive to the bundle orientation.
- The coefficient homomorphism $H^n(M; \mathbb{Z}) \longrightarrow H^n(M; \mathbb{Z}_2)$ carries the Euler class $e(E)$ to the top Stiefel-Whitney class $w_n(E)$. Thus, one can understand the top Stiefel-Whitney class as "the Euler class, ignoring orientation".
- If the vector bundle E possesses a nowhere zero global section, then $e(E) = 0$. This is obviously related to the second property of the Stiefel-Whitney classes.

Proofs for all these statements can be found in [MS74, §9]. Given an oriented smooth even-dimensional manifold M, the Euler class can be constructed in terms of the Čech-de Rham complex, i.e. it can be constructed as a differential form by the general Chern-Weil approach, such that via the identity

$$\chi(M) = \int_M e(\mathrm{T}M)$$

the Euler characteristic $\chi(M)$ and the Euler class (rather the Euler number) of the tangent bundle $e(\mathrm{T}M)$ are related, cf. [BT82, §14].

3.7. Chern classes

The Chern classes are essentially the complex pendant to the Stiefel-Whitney classes. Accordingly, their defining axioms are almost identical to those of the Stiefel-Whitney classes:

[e]In fact, the Euler characteristic is a **homotopy invariant**, i.e. if two spaces M, N have the same homotopy groups $\pi_i(M)$ and $\pi_i(N)$, their Euler characteristics $\chi(M)$ and $\chi(N)$ is equal—this is much stronger than just topological invariance. [Hat02, chp. 4] provides a geometrically motivated introduction to the subject.

(1) For every complex bundle $E \xrightarrow{\pi} M$ there corresponds a sequence of cohomology classes
$$c_i(E) \in H^{2i}(M; \mathbb{Z}),$$
called the **Chern classes** of E. The zeroth class $c_0(E)$ is equal to the unit element of the group $H^0(M; \mathbb{Z})$, and $c_i(E) = 0$ for $i > n$ if $E \xrightarrow{\pi} M$ is a complex n-vector bundle.

(2) Let $E \xrightarrow{\pi} M$, be a complex vector bundle and $g : N \longrightarrow M$ a continuous mapping. If g^*E is the complex pullback vector bundle over N, then
$$c_i(g^*E) = g^*c_i(E),$$
which is formally the same as the Whitney product formula of sec. 3.5.

(3) Let $E \xrightarrow{\pi} M$, $E' \xrightarrow{\pi'} M$ be two complex vector bundles over the same base space, then there is a product formula
$$c_k(E \oplus E') = \sum_{i=0}^{k} c_i(E) \smile c_{k-i}(E').$$

(4) For the complex canonical line bundle $\gamma^1_{\mathbb{C}} \xrightarrow{\mathrm{pr}_1} \mathbb{C}\mathcal{P}^1$ (complex analogon of the Möbius strip) the first Chern class is a generator of $H^2(\mathbb{C}\mathcal{P}^1; \mathbb{Z}) \cong \mathbb{Z}$ specified in advance.

Proving that this axioms in fact uniquely define the Chern classes again requires considerable efford, cf. [Hat03, thm. 3.2], but along similar lines of reasoning as in the case of the Stiefel-Whitney classes. Furthermore, any complex n-vector bundle $E \xrightarrow{\pi} M$ can be regarded as a real $2n$-vector bundle $E_{\mathbb{R}} \xrightarrow{\pi} M$. The top Chern class $c_n(E)$ is equal to the Euler class $e(E_{\mathbb{R}})$ of the underlying real bundle, where n is the dimension of the bundle fibers.

Note that the Chern classes are cohomology groups with $\mathbb{Z}$-coefficients rather than $\mathbb{Z}_2$-coefficients as in the case of the Stiefel-Whitney classes. Thus, they naturally contain “more” or rather “stronger” information about the geometry and topology of the complex vector bundle, which can be emphasized as follows, cf. [Hat03, thm. 3.8]: Any odd Stiefel-Whitney class vanishes, i.e. $w_{2i+1}(E_{\mathbb{R}}) = 0$, and any even class $w_{2i}(E_{\mathbb{R}})$ is the image of $c_i(E)$ under the coefficient homomorphism
$$H^{2i}(M; \mathbb{Z}) \longrightarrow H^{2i}(M; \mathbb{Z}_2).$$
In particular, this implies $w_1(E_{\mathbb{R}}) = 0$ for any complex vector bundle $E \xrightarrow{\pi} M$, which perfectly corresponds to the fact that any complex n-manifold is (canonically) orientable when regarded as a real $2n$-manifold. By the same argument $w_3(E_{\mathbb{R}}) = 0$, i.e. for $n \geq 2$ any complex n-vector bundle has a spin$^{\mathbb{C}}$-structure. Thus, the existence of a spin structure for the complex vector bundle $E \xrightarrow{\pi} M$ is equivalent to the vanishing of its first Chern class.

Furthermore, there are the Pontryagin classes for real vector bundles. They are defined via the Chern classes by complexification of the considered real bundle. As those particular characteristic classes are not neccessary for the following chapters, the reader is referred to [MS74, §15] for any further information.

CHAPTER 4

Geometry of Spinors

Paul Dirac's invention of spinor theory for the theoretical description of the relativistic electron in the late 1920's stirred a mathematical reinvestigation of the Clifford algebras that were introduced in the mid 19th century. Thanks to the interplay of numerous mathematical disciplines—mostly differential geometry and algebraic topology—many results on spinors were proven that far surpassed Dirac's original ideas. The famous and cherished Atiyah-Singer index theorem can be seen as one ultimate result based on this theoretical framework.

In this chapter the mathematical properties of Clifford algebras, spin representations and spinor bundles are summarized, following the first two parts of the modern standard text [LM89] on the subject. In particular, the topological obstructions for obtaining spinor bundles are studied in detail. This makes use of the characteristic classes, particularly the Stiefel-Whitney classes, which were introduced in the last chapter.

4.1. Clifford algebras

The central concept in spinor theory is that of a Clifford algebra, which can be understood as a generalization of the exterior algebra. Let V be a vector space over the field F and $q : V \longrightarrow F$ a **quadratic form** on V, i.e. $q(\lambda v) = \lambda^2 q(v)$ holds for all $\lambda \in F$ and $v \in V$. Let

$$\mathcal{T}(V) := \bigoplus_{r=0}^{\infty} V^{\otimes r} = F \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

denote the **tensor algebra** of V, where addition is naturally given by vector space addition and multiplication is induced by tensor products. Define an ideal $\mathcal{I}_q(V) \subset \mathcal{T}(V)$ generated by all elements of the form $v \otimes v - q(v)1$ for $v \in V$.[a] The **Clifford algebra** associated to the F-vector space V with the quadratic form q is defined to be the quotient space

$$\mathrm{C}\ell(V, q) := \mathcal{T}(V)/\mathcal{I}_q(V).$$

There is a canonical embedding $V \hookrightarrow \mathrm{C}\ell(V, q)$ of the original vector space into the Clifford algebra, which provides the natural Clifford multiplication "$\cdot$" with vectors of V. The algebra $\mathrm{C}\ell(V, q)$ can be understood as generated by the vector space V subject to the relations $v \cdot v = q(v)1$. Using the polarization formula $2\langle v, w\rangle_q = q(v + w) - q(v) - q(w)$, which is valid for any quadratic form, a somewhat more accessible relation

$$v \cdot w + w \cdot v = 2\langle v, w\rangle_q$$

for $v, w \in V \subset \mathrm{C}\ell(V, q)$ is obtained. The fundamental relationship between the Clifford algebra $\mathrm{C}\ell(V, q)$ and the exterior algebra $\Lambda^* V$ is of utmost importance. Obviously, from the above relation it follows the canonical algebra isomorphism $\mathrm{C}\ell(V, 0) \cong \Lambda^* V$ for the trivial quadratic form $q = 0$. For general $q \neq 0$ there still exists a vector space isomorphism between the two algebras. Thus, the Clifford algebra is the natural generalization of the exterior algebra which also takes the additional structure of a given quadratic form into account.

Let $\alpha : V \longrightarrow V$ with $v \mapsto -v$ be the reflection mapping at the origin, then there is a unique extension of this map to the Clifford algebra, i.e. there exists a Clifford algebra mapping

[a] Some mathematical texts define the ideal $\mathcal{I}_q(V)$ to be generated by elements $v \otimes v + q(v)1$, which of course ultimately leads to the same conclusions by replacing $q \mapsto -q$. In particular, the important book [LM89] uses this sign modification.

$\tilde{\alpha} : \mathrm{C}\ell(V,q) \longrightarrow \mathrm{C}\ell(V,q)$ satisfying $\tilde{\alpha}|_V = \alpha$, sometimes called the **canonical automorphism**. In particular, the obvious property $\alpha^2 = \mathrm{Id}$ also extends to the Clifford algebra, thus inducing an eigenspace decomposition

$$\mathrm{C}\ell(V,q) = \mathrm{C}\ell^0(V,q) \oplus \mathrm{C}\ell^1(V,q),$$

which makes $\mathrm{C}\ell(V,q)$ into a $\mathbb{Z}_2$-graded algebra. $\mathrm{C}\ell^0(V,q)$ is called the **even** part and $\mathrm{C}\ell^1(V,q)$ the **odd** part of the Clifford algebra, as the elements $\varphi \in \mathrm{C}\ell^i(V,q)$ of each part can be written as $\varphi = v_1 \cdots v_r$ with either an even or odd number r of factors $v_i \in V \subset \mathrm{C}\ell(V,q)$.

4.2. Universal covering of the (special) orthogonal group

Albeit Clifford algebras have become an important tool in pure algebra, one of their greatest benefits is to provide explicit constructions for the universal coverings of the (special) orthogonal group. This ultimately gives rise to the spinor formalism used in physics.

The explicit construction of the universal covering groups is rather technical, but cannot be avoided for a full understanding. Let $\mathrm{O}(V,q) := \{\lambda \in \mathrm{GL}(V) : \lambda^* q = q\}$ denote the group of ***q*-orthogonal linear mappings**, i.e. mappings which preserve the chosen quadratic form on V. In case of the Euclidean quadratic form q_n, this simply is isomorphic to the ordinary group $\mathrm{O}(n)$. Likewise, the **special *q*-orthogonal group** is defined to be $\mathrm{SO}(V,q) := \{\lambda \in \mathrm{O}(V,q) : \det \lambda = 1\}$. This generalizes the definitions from sec. A.9 to non-Euclidean metrics. Let

$$\mathrm{C}\ell^\times(V,q) := \big\{\varphi \in \mathrm{C}\ell(V,q) : \text{there exists an inverse } \varphi^{-1} \text{ with } \varphi^{-1} \cdot \varphi = \varphi \cdot \varphi^{-1} = 1\big\}$$

be the **multiplicative group of units** in the Clifford algebra, which is a 2^n-dimensional Lie group for $n = \dim_F V$ and $F = \mathbb{R}$ or $\mathbb{C}$. The associated Lie algebra $\mathfrak{cl}^\times(V,q)$ corresponds to the original Clifford algebra, i.e. $\mathfrak{cl}^\times(V,q) = \mathrm{C}\ell(V,q)$, and the Lie bracket is given by the ordinary commutator $[x,y] = x \cdot y - y \cdot x$ using the Clifford multiplication. In particular, the subspace of vectors $v \in V$ satisfying $q(v) \neq 0$ is contained in $\mathrm{C}\ell^\times(V,q)$, as $v^{-1} = \frac{1}{q(v)} v$ gives the inverse elements with respect to the Clifford multiplication.

The conjugation (or adjoint) mapping $\mathrm{Ad}_\varphi(x) := \varphi \cdot x \cdot \varphi^{-1}$ for $\varphi \in \mathrm{C}\ell^\times(V,q)$ using the Clifford multiplication induces a group homomorphism

$$\begin{aligned} \mathrm{Ad} : \mathrm{C}\ell^\times(V,q) &\longrightarrow \mathrm{Aut}\big(\mathrm{C}\ell(V,q)\big) \\ \varphi &\mapsto \mathrm{Ad}_\varphi, \end{aligned}$$

called the **adjointment** mapping on the Clifford algebra. This should not be confused with an adjoint representation (which would be a mapping $\mathrm{C}\ell^\times(V,q) \longrightarrow \mathrm{Aut}\big(\mathfrak{cl}^\times(V,q)\big)$, albeit a number of books call it this way). In addition, by setting $\tilde{\mathrm{Ad}}_\varphi(x) := \tilde{\alpha}(\varphi) x \varphi^{-1}$ define a **twisted adjointment** mapping

$$\begin{aligned} \tilde{\mathrm{Ad}} : \mathrm{C}\ell^\times(V,q) &\longrightarrow \mathrm{Aut}\big(\mathrm{C}\ell(V,q)\big) \\ \varphi &\mapsto \tilde{\mathrm{Ad}}_\varphi \end{aligned}$$

is defined, i.e. the twist is caused by utilizing the Clifford extension of the α mapping (that was introduced to make $\mathrm{C}\ell(V,q)$ into a $\mathbb{Z}_2$-graded algebra). Then define

$$\tilde{\Upsilon}(V,q) := \big\{\varphi \in \mathrm{C}\ell^\times(V,q) : \tilde{\mathrm{Ad}}_\varphi(V) = V\big\}$$

to be the subgroup of $\mathrm{C}\ell^\times(V,q)$ which induces ***V*-stable twisted adjointment** mappings. This subgroup $\tilde{\Upsilon}(V,q)$ consists of those elements of the Clifford Lie group $\mathrm{C}\ell^\times(V,q)$, which induce twisted adjointment mappings such that the original vector space V is kept invariant (mapped into itself). Furthermore, define

$$\begin{aligned} \Upsilon(V,q) &:= \big\{\varphi \in \mathrm{C}\ell^\times(V,q) \text{ with } \varphi = v_1 \cdots v_r \text{ and } q(v_i) \neq 0 \text{ for } v_i \in V\big\}, \\ \mathrm{S}\Upsilon(V,q) &:= \Upsilon(V,q) \cap \mathrm{C}\ell^0(V,q) \end{aligned}$$

to be the subgroups of $\mathrm{C}\ell^{\times}(V,q)$ generated by the vector space elements $v \in V$ with $q(v) \neq 0$. It can be shown, that $\Upsilon(V,q) \subset \tilde{\Upsilon}(V,q)$ holds and $\tilde{\mathrm{Ad}}|_{\mathrm{S}\Upsilon(V,q)} = \mathrm{Ad}$ is obvious from the definition of the group $\mathrm{S}\Upsilon(V,q)$.

There are many geometric properties of the twisted and ordinary adjointment mapping, which cannot be discussed in more detail, see [LM89, §2]. The important point here is, that the restriction of Ad to $\Upsilon(V,q)$ and $\tilde{\mathrm{Ad}}$ to $\tilde{\Upsilon}(V,q)$ both yield a q-orthogonal representation of the respective subgroup on the space V, as shown in the following diagram:

$$\begin{tikzcd}
\mathrm{C}\ell^{\times}(V,q) \arrow[d, bend right, "\mathrm{Ad}"'] \arrow[d, bend left, "\tilde{\mathrm{Ad}}"] & \overset{\text{subgroup}}{\supset} & \tilde{\Upsilon}(V,q) \arrow[drr, dashed, "\tilde{\mathrm{Ad}}"'] & \overset{\text{subgroup}}{\supset} & \Upsilon(V,q) \arrow[d, "\tilde{\mathrm{Ad}}"'] & \overset{\text{subgroup}}{\supset} & \mathrm{S}\Upsilon(V,q) \arrow[d, "\tilde{\mathrm{Ad}}=\mathrm{Ad}"] \\
\mathrm{Aut}\left(\mathrm{C}\ell^{\times}(V,q)\right) & \Longrightarrow & \underset{\text{subgroup}}{\supset} & & \mathrm{O}(V,q) & \underset{\text{subgroup}}{\supset} & \mathrm{SO}(V,q)
\end{tikzcd}$$

This gives rise to the following definitions: The **Pin group** of the vector space V with quadratic form q is the subgroup $\mathrm{Pin}(V,q) \subset \Upsilon(V,q)$ generated by the elements of the generalized unit sphere, i.e.

$$\mathrm{Pin}(V,q) := \{\text{products } v_1 \cdots v_r \in \Upsilon(V,q) \text{ with } q(v_j) = \pm 1 \text{ for all } j\}.$$

The associated **Spin group** of (V,q) is defined to be the even part of the Pin group, such that it is in fact the corresponding subgroup of $\mathrm{S}\Upsilon(V,q)$, i.e.

$$\begin{aligned}
\mathrm{Spin}(V,q) &:= \mathrm{Pin}(V,q) \cap \mathrm{C}\ell^0(V,q) \\
&= \{\text{products } v_1 \cdots v_r \in \mathrm{Pin}(V,q) \text{ with even number } r \text{ of factors}\} \\
&= \{\text{products } v_1 \cdots v_r \in \mathrm{S}\Upsilon(V,q) \text{ with } q(v_j) = \pm 1 \text{ for all } j\}.
\end{aligned}$$

The important property of the Spin and Pin group is that they are the unique covering groups of the (special) orthogonal group, which requires some work to prove in detail, see [LM89, thm. I.2.9]. This is usually expressed by the fact, that the two short sequences

$$\begin{array}{l}
0 \longrightarrow \mathcal{F} \hookrightarrow \mathrm{Spin}(V,q) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(V,q) \longrightarrow 1 \\
0 \longrightarrow \mathcal{F} \hookrightarrow \mathrm{Pin}(V,q) \xrightarrow{\tilde{\mathrm{Ad}}} \mathrm{O}(V,q) \longrightarrow 1
\end{array}
\qquad \text{where } \mathcal{F} := \begin{cases} \mathbb{Z}_2 & : \sqrt{-1} \notin F \\ \mathbb{Z}_4 & : \text{otherwise} \end{cases}$$

are both exact, i.e. the first mapping is injective, the second mapping surjective. Thus, it follows at once $\mathrm{SO}(V,q) \cong \mathrm{Spin}(V,q)/\mathcal{F}$ and $\mathrm{O}(V,q) \cong \mathrm{Pin}(V,q)/\mathcal{F}$. It is interesting to note the dependence of the covering groups on the original vector space's field F, i.e. whether they are 2-sheeted or 4-sheeted coverings.

4.3. Real and complex Clifford algebras

Up to this point the treatment was kept completely general and coordinate-independent. However, in the case of real or complex vector spaces V one can consider the prototype vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ after a choice of basis, which greatly simplifies the further discussion. Define the **real Clifford algebras** $\mathrm{C}\ell_{r,s} := \mathrm{C}\ell(\mathbb{R}^{r+s}, q_{r,s})$ and $\mathrm{C}\ell_n := \mathrm{C}\ell_{n,0}$ using the quadratic form

$$q_{r,s}(x) := \underbrace{x_1^2 + \cdots + x_r^2}_{r \text{ times}} \underbrace{-x_{r+1}^2 - \cdots - x_{r+s}^2}_{s \text{ times}}.$$

Since all non-degenerate quadratic forms on $\mathbb{C}^n$ are equivalent, the complexification of the real Clifford algebra $\mathrm{C}\ell_{r,s} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{C}\ell(\mathbb{C}^{r+s}, q \otimes \mathbb{C})$ corresponds to the **complex Clifford algebras** $\mathbb{C}\ell_n := \mathrm{C}\ell(\mathbb{C}^n, q_n^{\mathbb{C}})$ with $n = r+s$ and the complex quadratic form

$$q_n^{\mathbb{C}}(z) := \sum_{i=1}^{n} z_j^2.$$

Obviously, $\dim_{\mathbb{R}} \mathrm{C}\ell_{r,s} = 2^{r+s}$ and $\dim_{\mathbb{C}} \mathbb{C}\ell_n = 2^n$. Furthermore, for the complex Clifford algebras this equivalence of the (complex) quadratic forms yields the very useful identity

$$\mathbb{C}\ell_n \cong \mathrm{C}\ell_{n,0} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{C}\ell_{n-1,1} \otimes_{\mathbb{R}} \mathbb{C} \cong \ldots \cong \mathrm{C}\ell_{0,n} \otimes_{\mathbb{R}} \mathbb{C}.$$

Due to the simple nature of those Clifford algebras they can be understood in the following manner, cf. [LM89, thm. I.3.1]: Let $e_1, \ldots, e_{r+s}$ be any $q_{r,s}$-orthonormal basis of $\mathbb{R}^{r+s} \subset \mathrm{C}\ell_{r,s}$. The associated Clifford algebra $\mathrm{C}\ell_{r,s}$ is then generated as an algebra by $r+s$ basis elements $e_1, \ldots, e_{r+s}$ subject to the relations

$$e_i e_j + e_j e_i = \begin{cases} +2 & : \quad i = j = 1, \ldots, r \\ -2 & : \quad i = j = r+1, \ldots, r+s \\ 0 & : \quad i \neq j \end{cases} .$$

Thus, a real Clifford algebra $\mathrm{C}\ell_{r,s}$ is generated by r linearly independent elements squaring to $+1$ and s linearly independent elements with square -1.[b] Using this description, the real and complex Clifford algebras can be understood as ordinary matrix algebras over real $\mathbb{R}$, complex $\mathbb{C}$ or quaternionic $\mathbb{H}$ numbers. Furthermore there are several periodicity and symmetry isomorphisms, closely related to the Bott periodicity[c] found in homotopy theory:

general isomorphisms:	$\mathrm{C}\ell_{n+2,0} \cong \mathrm{C}\ell_{0,n} \otimes \mathrm{C}\ell_{2,0}$	
	$\mathrm{C}\ell_{0,n+2} \cong \mathrm{C}\ell_{n,0} \otimes \mathrm{C}\ell_{0,2}$	
	$\mathrm{C}\ell_{r+1,s+1} \cong \mathrm{C}\ell_{r,s} \otimes \mathrm{C}\ell_{1,1}$	
periodicity isomorphisms:	$\mathrm{C}\ell_{n+8,0} \cong \mathrm{C}\ell_{n,0} \otimes \mathrm{C}\ell_{8,0}$	$\mathbb{C}\ell_{n+2} \cong \mathbb{C}\ell_n \otimes_{\mathbb{C}} \mathbb{C}\ell_2$
	$\mathrm{C}\ell_{0,n+8} \cong \mathrm{C}\ell_{0,n} \otimes \mathrm{C}\ell_{0,8}$	
symmetry isomorphisms:	$\mathrm{C}\ell_{r,s} \cong \mathrm{C}\ell_{r-4,s+4} \cong \mathrm{C}\ell_{r+4,s-4}$	
	$\mathrm{C}\ell_{r,s} \cong \mathrm{C}\ell_{s+1,r-1}$	
even part isomorphisms:	$\mathrm{C}\ell^0_{r,s} \cong \mathrm{C}\ell_{r,s-1}$	

where $\mathrm{C}\ell_{8,0} \cong \mathrm{C}\ell_{0,8} \cong \mathbb{R}(16)$ and $\mathbb{C}\ell_2 \cong \mathbb{C}(2)$. Obviously, it suffices to know only a few Clifford algebras $\mathrm{C}\ell_{r,s}$ to get the entire range, see [LM89, p. 29] or tab. 4.1.

The short exact sequences for the orthogonal groups mentioned above now specialize to

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2 \arrow[r, hook] & \mathrm{Spin}(r,s) \arrow[r, two heads] & \mathrm{SO}(r,s) \arrow[r] & 1 \\ 0 \arrow[r] & \mathbb{Z}_2 \arrow[r, hook] & \mathrm{Pin}(r,s) \arrow[r, two heads] & \mathrm{O}(r,s) \arrow[r] & 1 \end{tikzcd}$$

for all r and s. Thus, both $\mathrm{Pin}(r,s)$ and $\mathrm{Spin}(r,s)$ are the unique 2-sheeted universal covering Lie groups of the (special) orthogonal groups $\mathrm{O}(r,s)$ and $\mathrm{SO}(r,s)$.

4.4. Representation of the Clifford algebra and chirality splitting

After the discussion of general properties of Clifford algebras, the representation theory remains to be investigated. Let V be a F-vector space, $q : V \longrightarrow F$ a quadratic form and $\mathrm{C}\ell(V,q)$ the associated Clifford algebra. Furthermore, let F be contained in the field $\tilde{F} \supset F$ and W be a vector space over the field $\tilde{F}$. A **$\tilde{F}$-representation of the Clifford algebra** $\mathrm{C}\ell(V,q)$ is a F-algebra homomorphism

$$\rho : \mathrm{C}\ell(V,q) \longrightarrow \mathrm{End}_{\tilde{F}}(W)$$

that is conveniently denoted as $\varphi \cdot w := \rho_\varphi(w) = \big[\rho(\varphi)\big](w)$, effectively making the representation space W into a $\mathrm{C}\ell(V,q)$-module.

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[b] Obviously, for the quadratic form induced by the Minkowski inner product on $\mathbb{R}^4$ this gives the algebraic relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ which define the Dirac γ-matrices.

[c] The Bott periodicity theorem is a deep result from homotopy theory proved in the 1950s, which relates the homotopy of $\mathrm{O}(n)$ or $\mathrm{U}(n)$ with the homotopy of their respective loop spaces $\Omega\,\mathrm{O}$ or $\Omega\,\mathrm{U}$. This leads to the periodicity results $\pi_i\,\mathrm{O} \cong \pi_{i+8}\,\mathrm{O}$ and $\pi_i\,\mathrm{U} \cong \pi_{i+2}\,\mathrm{U}$, cf. [Mil63, thm. 24.7].

	$t=$							**real Clifford algebras $\mathrm{C}\boldsymbol{\ell}_{r,s}$**								$\mathbb{C}\boldsymbol{\ell}_n$
	$+7$	$+6$	$+5$	$+4$	$+3$	$+2$	$+1$	± 0	-1	-2	-3	-4	-5	-6	-7	
$n=0$								$\mathbb{R}$								$\mathbb{C}$
1							${}^2\mathbb{R}$		$\mathbb{C}$							${}^2\mathbb{C}$
2						$\mathbb{R}(2)$		$\mathbb{R}(2)$		$\mathbb{H}$						$\mathbb{C}(2)$
3					$\mathbb{C}(2)$		${}^2\mathbb{R}(2)$		$\mathbb{C}(2)$		${}^2\mathbb{H}$					${}^2\mathbb{C}(2)$
4				$\mathbb{H}(2)$		$\mathbb{R}(4)$		$\mathbb{R}(4)$		$\mathbb{H}(2)$		$\mathbb{H}(2)$				$\mathbb{C}(4)$
5			${}^2\mathbb{H}(2)$		$\mathbb{C}(4)$		${}^2\mathbb{R}(4)$		$\mathbb{C}(4)$		${}^2\mathbb{H}(2)$		$\mathbb{C}(4)$			${}^2\mathbb{C}(4)$
6		$\mathbb{H}(4)$		$\mathbb{H}(4)$		$\mathbb{R}(8)$		$\mathbb{R}(8)$		$\mathbb{H}(4)$		$\mathbb{H}(4)$		$\mathbb{R}(8)$		$\mathbb{C}(8)$
7	$\mathbb{C}(8)$		${}^2\mathbb{H}(4)$		$\mathbb{C}(8)$		${}^2\mathbb{R}(8)$		$\mathbb{C}(8)$		${}^2\mathbb{H}(4)$		$\mathbb{C}(8)$		${}^2\mathbb{R}(8)$	${}^2\mathbb{C}(8)$

TABLE 4.1. Matrix representations of the real and complex Clifford algebras. In case of the real Clifford algebras $\mathrm{C}\ell_{r,s}$ the sum $n = r + s$ denotes the dimension of the underlying vector space and $t = r - s$ the space-time signature, i.e. the shift of the indices. Read $\mathbb{R}(4) = \mathbb{R}^{4\times 4} = \mathrm{Mat}_4(\mathbb{R})$ and an upper prefix "2" as a double sum, e.g. ${}^2\mathbb{H}(2) = \mathrm{Mat}_2(\mathbb{H}) \oplus \mathrm{Mat}_2(\mathbb{H})$.

Like for any other kind of representation, the question of irreducibility is of particular importance. Let $\nu_{r,s}$ denote the number of inequivalent irreducible real representations of $\mathrm{C}\ell_{r,s}$ and $\nu_n^{\mathbb{C}}$ the inequivalent irreducible complex representations of $\mathbb{C}\ell_n$, then

$$\nu_{r,s} = \begin{cases} 2 & : \quad s+1-r \equiv 0 \bmod 4 \\ 1 & : \quad \text{otherwise} \end{cases} \qquad \text{and} \qquad \nu_n^{\mathbb{C}} = \begin{cases} 2 & : \quad n \text{ odd} \\ 1 & : \quad n \text{ even} \end{cases}$$

as proved in [LM89, thm. I.5.7]. If $d_n := \dim_{\mathbb{R}} W$ denotes the real dimension of the corresponding irreducible $\mathrm{C}\ell_n$-module and $K_n = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ the maximal commuting subalgebra (and analogous for the complexified $\mathbb{C}\ell_n$-modules), a complete workout of the relevant Clifford algebras yields the data shown in tab. 4.2.

Another important aspect is the chirality splitting of the Clifford algebras in certain dimensions, which essentially determines the inequivalent irreducible representations. This should not be confused with the even / odd splitting $\mathrm{C}\ell(V,q) = \mathrm{C}\ell^0(V,q) \oplus \mathrm{C}\ell^1(V,q)$, however, it is constructed in the same fashion. Let $e_1, \ldots, e_{r+s}$ be the Clifford algebra generators of $\mathrm{C}\ell_{r,s}$. The product

$$\omega := e_1 \cdots e_{r+s}$$

is called the **volume element** of the algebra. In the complex case there is a corresponding element $\omega_{\mathbb{C}} \in \mathbb{C}\ell_n$ by setting

$$\omega_{\mathbb{C}} := \mathrm{i}^{\lfloor \frac{n+1}{2} \rfloor} \omega,$$

which is called the **complex volume element**. Of course, both definitions are independent of the choice of basis elements. Note the equality $\omega_{\mathbb{C}} = \omega$ in dimensions 7, 8 modulo 8. For n odd both ω and $\omega_{\mathbb{C}}$ are central elements of the respective algebra, i.e. $\omega \cdot \varphi = \varphi \cdot \omega$ for all $\varphi \in \mathrm{C}\ell_{r,s}$ and $\omega_{\mathbb{C}} \cdot \psi = \psi \cdot \omega_{\mathbb{C}}$ for all $\psi \in \mathbb{C}\ell_n$. The volume elements then square according to

$$\omega^2 = (-1)^{\frac{n(n+1)}{2}+r} = \begin{cases} (-1)^r & : \quad n \equiv 3,4 \bmod 4 \\ (-1)^{r+1} & : \quad n \equiv 1,2 \bmod 4 \end{cases} \qquad \text{for } n := r+s,$$

$$(\omega_{\mathbb{C}})^2 = 1 \qquad \text{for all } n.$$

In order to achieve an eigenspace decomposition, the volume elements must both be central and satisfy $\omega^2 = 1$, i.e. n odd and $\omega^2 = 1$ are the conditions for an chirality splitting in the

n	mostly minus	ν	d	K	mostly plus	ν	d	K	$\mathbb{C}\ell_n$	$\nu^{\mathbb{C}}$	$d^{\mathbb{C}}$	$K^{\mathbb{C}}$
1	$\mathrm{C}\ell_{1,0} \cong \mathrm{C}\ell_1 \cong {}^2\mathbb{R}$	2	1	$\mathbb{R}$	$\mathrm{C}\ell_{0,1} \cong \mathbb{C}$	1	1	$\mathbb{C}$	${}^2\mathbb{C}$	2	1	$\mathbb{C}$
2	$\mathrm{C}\ell_{1,1} \cong \mathrm{C}\ell_2 \cong \mathbb{R}(2)$	1	2	$\mathbb{R}$	$\mathrm{C}\ell_{1,1} \cong \mathbb{R}(2)$	1	2	$\mathbb{R}$	$\mathbb{C}(2)$	1	2	$\mathbb{C}$
3	$\mathrm{C}\ell_{1,2} \cong \mathrm{C}\ell_3 \cong \mathbb{C}(2)$	1	4	$\mathbb{C}$	$\mathrm{C}\ell_{2,1} \cong {}^2\mathbb{R}(2)$	2	2	$\mathbb{R}$	${}^2\mathbb{C}(2)$	2	2	$\mathbb{C}$
4	$\mathrm{C}\ell_{1,3} \cong \mathrm{C}\ell_4 \cong \mathbb{H}(2)$	1	8	$\mathbb{H}$	$\mathrm{C}\ell_{3,1} \cong \mathbb{R}(4)$	1	4	$\mathbb{R}$	$\mathbb{C}(4)$	1	4	$\mathbb{C}$
5	$\mathrm{C}\ell_{1,4} \cong \mathrm{C}\ell_5 \cong {}^2\mathbb{H}(2)$	2	8	$\mathbb{H}$	$\mathrm{C}\ell_{4,1} \cong \mathbb{C}(4)$	1	8	$\mathbb{C}$	${}^2\mathbb{C}(4)$	2	4	$\mathbb{C}$
6	$\mathrm{C}\ell_{1,5} \cong \mathrm{C}\ell_6 \cong \mathbb{H}(4)$	1	16	$\mathbb{H}$	$\mathrm{C}\ell_{5,1} \cong \mathbb{H}(4)$	1	16	$\mathbb{H}$	$\mathbb{C}(8)$	1	8	$\mathbb{C}$
7	$\mathrm{C}\ell_{1,6} \cong \mathrm{C}\ell_7 \cong \mathbb{C}(8)$	1	16	$\mathbb{C}$	$\mathrm{C}\ell_{6,1} \cong {}^2\mathbb{H}(4)$	2	16	$\mathbb{H}$	${}^2\mathbb{C}(8)$	2	8	$\mathbb{C}$
8	$\mathrm{C}\ell_{1,7} \cong \mathrm{C}\ell_8 \cong \mathbb{R}(16)$	1	16	$\mathbb{R}$	$\mathrm{C}\ell_{7,1} \cong \mathbb{H}(8)$	1	32	$\mathbb{H}$	$\mathbb{C}(16)$	1	16	$\mathbb{C}$
9	$\mathrm{C}\ell_{1,8} \cong \mathrm{C}\ell_9 \cong {}^2\mathbb{R}(16)$	2	16	$\mathbb{R}$	$\mathrm{C}\ell_{8,1} \cong \mathbb{C}(16)$	1	32	$\mathbb{C}$	${}^2\mathbb{C}(16)$	2	16	$\mathbb{C}$
10	$\mathrm{C}\ell_{1,9} \cong \mathrm{C}\ell_{10} \cong \mathbb{R}(32)$	1	32	$\mathbb{R}$	$\mathrm{C}\ell_{9,1} \cong \mathbb{R}(32)$	1	32	$\mathbb{R}$	$\mathbb{C}(32)$	1	32	$\mathbb{C}$
11	$\mathrm{C}\ell_{1,10} \cong \mathrm{C}\ell_{11} \cong \mathbb{C}(32)$	1	64	$\mathbb{C}$	$\mathrm{C}\ell_{10,1} \cong {}^2\mathbb{R}(32)$	2	32	$\mathbb{R}$	${}^2\mathbb{C}(32)$	2	32	$\mathbb{C}$

TABLE 4.2. Classification table of Clifford algebras for all physically relevant dimensions, constructed from [LM89, p. 33] and tab. 4.1 using the numerous isomorphisms.

real case. The eigenspace decompositions of the Clifford algebras are then

$$\begin{aligned}
\mathrm{C}\ell_{r,s} &= \mathrm{C}\ell_{r,s}^+ \oplus \mathrm{C}\ell_{r,s}^- && \text{for } \omega^2 = 1 \text{ and } n \text{ odd} && && \mathrm{C}\ell_{r,s}^\pm = (1 \pm \omega)\,\mathrm{C}\ell_{r,s} \\
\mathrm{C}\ell_n &= \mathrm{C}\ell_n^+ \oplus \mathrm{C}\ell_n^- && \text{for } n \equiv 1 \bmod 4 && \text{where} && \mathrm{C}\ell_n^\pm = (1 \pm \omega)\,\mathrm{C}\ell_n \\
\mathbb{C}\ell_n &= \mathbb{C}\ell_n^+ \oplus \mathbb{C}\ell_n^- && \text{for } n \text{ odd} && && \mathbb{C}\ell_n^\pm = (1 \pm \omega_{\mathbb{C}})\mathbb{C}\ell_n
\end{aligned}$$

This corresponds exactly to the chirality decomposition induced by the γ_5 matrix in the Dirac spinor formalism used in physics. Note that the mapping $\tilde{\alpha}$ introduced for the even / odd splitting of the Clifford algebra actually interchanges the chirality eigenspaces $\mathrm{C}\ell_{r,s}^+$ and $\mathrm{C}\ell_{r,s}^-$.

It remains to determine how this chirality splitting affects the representations of the Clifford algebras. Let $\rho : \mathrm{C}\ell_{r,s} \longrightarrow \mathrm{End}_{\mathbb{R}}(W)$ be an irreducible representation of the real Clifford algebra $\mathrm{C}\ell_{r,s}$ with two inequivalent irreducible representations, see fig. 4.2 for a listing of low-dimensional Clifford algebras with Minkowskian signature. Then either of the possibilities

$$\rho(\omega) = \mathrm{Id} \quad \text{or} \quad \rho(\omega) = -\,\mathrm{Id}$$

can occur, and the corresponding representations are inequivalent. Thus, the representation of the volume element distinguishes the two inequivalent real representations of $\mathrm{C}\ell_{r,s}$. Of course, the analogous statement is also true for the complex Clifford algebra $\mathbb{C}\ell_n$ provided n is odd.

This chirality splitting is important as follows: For $n \equiv 1 \bmod 4$ there are two inequivalent real spin representations as explained in the previous paragraph. For $n \equiv 2 \bmod 4$ one can now consider the splitting

$$W = W^+ \oplus W^- \quad \text{where} \quad W^\pm = \big(1 \pm \rho(\omega)\big)W.$$

Then each of the subspaces W^+ and W^- is invariant under the even subalgebra $\mathrm{C}\ell_n^0$, and under the general isomorphism $\mathrm{C}\ell_n^0 \cong \mathrm{C}\ell_{n-1}$ these spaces correspond to the distinct irreducible real representations of $\mathrm{C}\ell_{n-1}$. Again, there are corresponding statements for $\mathrm{C}\ell_{r,s}$ and $\mathbb{C}\ell_n$.

4.5. Real and complex spin representations

Since $\mathrm{Spin}(r,s)$ and $\mathrm{Pin}(r,s)$ are Lie subgroups of both the real and complex Clifford algebras by the inclusions

$$\begin{aligned}
&\mathrm{Spin}(r,s) \subset \mathrm{Pin}(r,s) \subset \mathrm{C}\ell_{r,s} \\
&\mathrm{Spin}(r,s) \subset \mathrm{C}\ell_{r,s}^0 \subset \mathrm{C}\ell_{r,s},
\end{aligned}$$

n	**spin group isomorphisms**	**Clifford algebra**	ν	d	K
1	$\mathrm{Spin}(1,0) \subset \mathrm{C}\ell^0_{1,0} \cong \mathrm{C}\ell_0 \cong \mathrm{C}\ell^0_{0,1} \supset \mathrm{Spin}(0,1)$	$\mathrm{C}\ell_0 \cong \mathbb{R}$	1	1	$\mathbb{R}$
2	$\mathrm{Spin}(1,1) \subset \mathrm{C}\ell^0_{1,1} \cong \mathrm{C}\ell_1 \cong \mathrm{C}\ell^0_{1,1} \supset \mathrm{Spin}(1,1)$	$\mathrm{C}\ell_1 \cong {}^2\mathbb{R}$	2	1	$\mathbb{R}$
3	$\mathrm{Spin}(1,2) \subset \mathrm{C}\ell^0_{1,2} \cong \mathrm{C}\ell_2 \cong \mathrm{C}\ell^0_{2,1} \supset \mathrm{Spin}(2,1)$	$\mathrm{C}\ell_2 \cong \mathbb{R}(2)$	1	2	$\mathbb{R}$
4	$\mathrm{Spin}(1,3) \subset \mathrm{C}\ell^0_{1,3} \cong \mathrm{C}\ell_3 \cong \mathrm{C}\ell^0_{3,1} \supset \mathrm{Spin}(3,1)$	$\mathrm{C}\ell_3 \cong \mathbb{C}(2)$	1	4	$\mathbb{C}$
5	$\mathrm{Spin}(1,4) \subset \mathrm{C}\ell^0_{1,4} \cong \mathrm{C}\ell_4 \cong \mathrm{C}\ell^0_{4,1} \supset \mathrm{Spin}(4,1)$	$\mathrm{C}\ell_4 \cong \mathbb{H}(2)$	1	8	$\mathbb{H}$
6	$\mathrm{Spin}(1,5) \subset \mathrm{C}\ell^0_{1,5} \cong \mathrm{C}\ell_5 \cong \mathrm{C}\ell^0_{5,1} \supset \mathrm{Spin}(5,1)$	$\mathrm{C}\ell_5 \cong {}^2\mathbb{H}(2)$	2	8	$\mathbb{H}$
7	$\mathrm{Spin}(1,6) \subset \mathrm{C}\ell^0_{1,6} \cong \mathrm{C}\ell_6 \cong \mathrm{C}\ell^0_{6,1} \supset \mathrm{Spin}(6,1)$	$\mathrm{C}\ell_6 \cong \mathbb{H}(4)$	1	16	$\mathbb{H}$
8	$\mathrm{Spin}(1,7) \subset \mathrm{C}\ell^0_{1,7} \cong \mathrm{C}\ell_7 \cong \mathrm{C}\ell^0_{7,1} \supset \mathrm{Spin}(7,1)$	$\mathrm{C}\ell_7 \cong \mathbb{C}(8)$	1	16	$\mathbb{C}$
9	$\mathrm{Spin}(1,8) \subset \mathrm{C}\ell^0_{1,8} \cong \mathrm{C}\ell_8 \cong \mathrm{C}\ell^0_{8,1} \supset \mathrm{Spin}(8,1)$	$\mathrm{C}\ell_8 \cong \mathbb{R}(16)$	1	16	$\mathbb{R}$
10	$\mathrm{Spin}(1,9) \subset \mathrm{C}\ell^0_{1,9} \cong \mathrm{C}\ell_9 \cong \mathrm{C}\ell^0_{9,1} \supset \mathrm{Spin}(9,1)$	$\mathrm{C}\ell_9 \cong {}^2\mathbb{R}(16)$	2	16	$\mathbb{R}$
11	$\mathrm{Spin}(1,10) \subset \mathrm{C}\ell^0_{1,10} \cong \mathrm{C}\ell_{10} \cong \mathrm{C}\ell^0_{10,1} \supset \mathrm{Spin}(10,1)$	$\mathrm{C}\ell_{10} \cong \mathbb{R}(32)$	1	32	$\mathbb{R}$

TABLE 4.3. Spin group isomorphisms and classification of the corresponding even real Clifford algebras.

the representations of the Clifford algebras restrict to representations of both $\mathrm{Pin}(r,s)$ and $\mathrm{Spin}(r,s)$. In this section, the properties of those restricted representations are investigated, particularly the question of irreducibility.

Let $\rho_{r,s} : \mathrm{C}\ell_{r,s} \longrightarrow \mathrm{End}_{\tilde{F}}(S)$ be an irreducible $\tilde{F}$-representation of the real Clifford algebra. The restriction of this mapping to the $\mathrm{O}(r,s)$-covering group $\mathrm{Pin}(r,s) \subset \mathrm{C}\ell_{r,s}$ is called a **pinor representation**, which is also irreducible and of the same type (real, complex, quaternionic) as the Clifford algebra's matrix algebra, i.e. of the type indicated by K in tab. 4.2. It is of utmost importance to realize that the pinor groups and representations are dependent of the sign of the space-time signature, i.e. $\mathrm{Pin}(r,s) \not\cong \mathrm{Pin}(s,r)$, which is obvious from the properties of the Clifford algebras.

This is not the case for the representations of the spin group, as the isomorphisms of sec. 4.3 imply

$$\mathrm{C}\ell^0_{r,s} \cong \mathrm{C}\ell_{r,s-1} \cong \mathrm{C}\ell_{s,r-1} \cong \mathrm{C}\ell^0_{s,r}$$

for the even part of the Clifford algebras, and this also descents to the spin groups, i.e. in general $\mathrm{Spin}(r,s) \cong \mathrm{Spin}(s,r)$ holds, see tab. 4.3. Let $\rho_{r,s} : \mathrm{C}\ell_{r,s} \longrightarrow \mathrm{End}_{\tilde{F}}(S)$ be an irreducible $\tilde{F}$-representation of the Clifford algebra $\mathrm{C}\ell_{r,s}$, this induces a **spinor representation**

$$\Delta_{r,s} := \rho_{r,s}\big|_{\mathrm{Spin}(r,s)} : \mathrm{Spin}(r,s) \longrightarrow \mathrm{GL}(S)$$

by restriction to $\mathrm{Spin}(r,s) \subset \mathrm{C}\ell_{r,s}$. However, unlike for the pinor representations, a spinor representation obtained in this way may not be irreducible. In general, an irreducible pinor representation is a direct sum of either one or two irreducible spinor representations. Again, similar statements are available for the complex case.

In the physical context, pinor representations depend on the full Lorentz group, whereas spinor representations only rely on the proper orthochronous subgroup, see sec. 4.8. The properties of the spinor representations obtained by restricting representations of Clifford algebras in the case of a Minkowskian signature can be summarized as follows (cf. tab. 4.2 and tab. 4.3):

(1) A **real spinor representation** of the universal $\mathrm{SO}(r,s)$-covering group $\mathrm{Spin}(r,s)$ on the vector space S is a homomorphism

$$\Delta_{r,s} : \mathrm{Spin}(r,s) \longrightarrow \mathrm{GL}(S)$$

induced by restricting an irreducible representation of the real Clifford algebra $\mathrm{C}\ell_{r,s}$ to $\mathrm{Spin}(r,s) \subset \mathrm{C}\ell^0_{r,s} \subset \mathrm{C}\ell_{r,s}$. Note that "real" does not refer to a real representation (see this issue below) but to the usage of a real Clifford algebra.

The real spinor representations are fully understood using the Bott periodicity. Let $\Delta_d : \mathrm{Spin}(1, d-1) \longrightarrow \mathrm{GL}(S)$ be a real spinor representation induced from a $\mathrm{C}\ell_{1,d-1}$-representation, then one of the following cases holds:

- $d \equiv 1, 5 \bmod 8$: The spinor representation Δ_d is independent of the used irreducible representation of the Clifford algebra $\mathrm{C}\ell_{1,d-1}$.
- $d \equiv 2, 6 \bmod 8$: There is a decomposition $\Delta_{4m+2} = \Delta^+_{4m+2} \oplus \Delta^-_{4m+2}$ of the spinor representation, where Δ^+_{4m+2} and Δ^-_{4m+2} are the inequivalent irreducible representations of $\mathrm{Spin}(1, 4m+1)$.
- $d \equiv 3, 4 \bmod 8$: The spinor representation Δ_d is a direct sum of two equivalent irreducible representations.
- $d \equiv 7, 8 \bmod 8$: The spinor representation Δ_d is irreducible.

Most of these properties follow from $\mathrm{Spin}(1, d-1) \subset \mathrm{C}\ell^0_{1,d-1} \cong \mathrm{C}\ell_{d-1}$ and the corresponding properties of $\mathrm{C}\ell_{d-1}$, cf. tab. 4.2. In particular, the type of the representation (real, complex or quaternionic) is determined from the maximal commuting subalgebra of $\mathrm{C}\ell^0_{1,d-1} \cong \mathrm{C}\ell_{d-1}$, which is denoted by K in tab. 4.2. Thus,

- $n \equiv 1, 2, 3 \bmod 8$: The representation Δ_d is real.
- $n \equiv 4, 8 \bmod 8$: The representation Δ_d is complex.
- $n \equiv 5, 6, 7 \bmod 8$ The representation Δ_d is quaternionic.

All those properties of the real spinor representations are again summarized in the table on p. 63. Obviously, the real representations of the spin groups are rather complicated. To circumvent this, physics is usually formulated in terms of the complex spinor representations, which stem from the representations of the complex Clifford algebras. Here all the signature ambiguities vanish, and the 8-periodicity of the real spinor representations is reduced to a much simpler 2-periodicity.

(2) A **complex spin representation** of the group $\mathrm{Spin}(1, d-1)$ is a homomorphism

$$\Delta^{\mathbb{C}}_d : \mathrm{Spin}(1, d-1) \longrightarrow \mathrm{GL}_{\mathbb{C}}(S)$$

induced by restricting an irreducible complex representation $\rho_d : \mathbb{C}\ell_d \longrightarrow \mathrm{End}_{\mathbb{C}}(S)$ of the complex Clifford algebra $\mathbb{C}\ell_d$ to $\mathrm{Spin}(1, d-1) \subset \mathrm{C}\ell^0_{1,d-1} \subset \mathbb{C}\ell_d$.

Using the Bott periodicity theorem again, the complex spinor representations can be understood in the same terms as the real ones. Given a complex spinor representation $\Delta^{\mathbb{C}}_d : \mathrm{Spin}(1, d) \longrightarrow \mathrm{GL}_{\mathbb{C}}(S)$, either one of the following two cases holds:

- $d \equiv 1 \bmod 2$ (d odd): The complex spinor representation $\Delta^{\mathbb{C}}_d$ is independent of the used irreducible representation of the complex Clifford algebra $\mathbb{C}\ell_d$.
- $d \equiv 2 \bmod 2$ (d even): There is a decomposition $\Delta^{\mathbb{C}}_{2m} = \Delta^{\mathbb{C}+}_{2m} \oplus \Delta^{\mathbb{C}-}_{2m}$ of the spinor representation, where $\Delta^{\mathbb{C}+}_{2m}$ and $\Delta^{\mathbb{C}-}_{2m}$ are the inequivalent irreducible complex representations of $\mathrm{Spin}(2m)$.

Since $K^{\mathbb{C}}_d = \mathbb{C}$ for all dimensions d, all complex spin representations are in fact complex representations—just as the name suggests.

Several properties (see [LM89, p. 39] or [BtD85, IV.§6] for details) relate the real and complex spinor representations in a nontrivial manner. But in the case $d = 8m+2$ the complex spinor representation emerges from the real one by simple complexification, i.e.

$$\Delta^{\pm}_{8m+2} \otimes \mathbb{C} \cong \Delta^{\mathbb{C}\pm}_{8m+2}.$$

At this point the similarities to the spinors used in physics are striking already. Nevertheless this only establishes those properties within a single Clifford algebra and not in an entire spinor field as implied in physics. Thus, one has to apply the concept of fiber and vector bundles again and specialize to spinor and Dirac bundles.

4.6. Orientability and spin structures on vector bundles

A real vector bundle $E \xrightarrow{\pi} M$ with n-dimensional fibers (i.e. E looks locally like $U \times \mathbb{R}^n$) can always be equipped with a **Riemannian** or **Euclidean structure**, which provides a positive definite inner product continuously defined in the fibers. More general, a **pseudo-Riemannian structure** provides an arbitrary inner product (i.e. not necessarily positive definite) in every fiber. The corresponding object for a complex vector bundle is called a **Hermitian structure**. If the associated determinant line bundle

$$K_E := \Lambda^n E \xrightarrow{\tilde{\pi}} M$$

admits a global, nowhere vanishing section, the bundle is called **orientable**. Note that such a section has to be global, which is a very strong restriction—the Möbius strip (viewed as a non-trivial real 1-dimensional vector bundle over the base space S^1, see fig. 4.1) and the Klein bottle (viewed as a non-trivial S^1 fiber bundle over S^1, see fig. 4.1) are simple examples of non-orientable bundles, whereas a double-twisted band is orientable. It remains to determine if a given vector bundle $E \xrightarrow{\pi} M$ is orientable. This question is answered by the first Stiefel-Whitney cohomology class $w_1(E) \in H^1(X; \mathbb{Z}_2)$, which was introduced in the previous chapter.

As explained in sec. 2.2, for each n-dimensional vector bundle $E \xrightarrow{\pi} M$ there is a corresponding frame bundle, i.e. a principal $\mathrm{GL}_n(\mathbb{R})$-bundle $P_{\mathrm{GL}}(E) \xrightarrow{\tilde{\pi}} M$. Furthermore, there is a general procedure to reduce the structure group of the frame bundle, as outlined in [Hus98, chp. 6]. Given a Riemannian structure on a vector bundle, the structure group can be reduced to $\mathrm{O}(n) \subset \mathrm{GL}_n(\mathbb{R})$ due to the preservation of the inner product. If the bundle is orientable, it can be reduced even further to $\mathrm{SO}(n) \subset \mathrm{O}(n) \subset \mathrm{GL}_n(\mathbb{R})$, preserving orientation. Since $\mathrm{SO}(n)$ has a standard representation on the vector space $\mathbb{R}^n$, the associated vector bundle $H := P_{\mathrm{SO}}(E) \times_{\mathrm{SO}} \mathbb{R}^n \xrightarrow{\tilde{\pi}} M$ is well-defined. But this is just a real n-dimensional vector bundle with inner structure group $\mathrm{SO}(n)$, i.e. an orientable vector bundle with Riemannian structure, and one can prove that H is in fact isomorphic to E. Following similar lines of reasoning, there is a bunch of equivalence relations between vector bundles with additional structure and principal bundles with subgroups of the general linear group:

$$\begin{tikzcd} \begin{array}{l}\text{real vector bundle}\\ E \xrightarrow{\pi} M\end{array} \arrow[r, bend left, "\text{frame bundle}"] & \begin{array}{l}\text{principal } \mathrm{GL}_n(\mathbb{R})\text{-bundle}\\ P \xrightarrow{\pi} M\end{array} \arrow[l, bend left, "\text{associated bundle}"] \end{tikzcd}$$

$$\begin{array}{rcl}
\text{real } n\text{-vector bundle} & \Longleftrightarrow & \text{principal } \mathrm{GL}_n(\mathbb{R})\text{-bundle} \\
\text{real oriented } n\text{-vector bundle} & \Longleftrightarrow & \text{principal } \mathrm{GL}_n^+(\mathbb{R})\text{-bundle} \\
\text{real Riemannian } n\text{-vector bundle} & \Longleftrightarrow & \text{principal } \mathrm{O}(n)\text{-bundle} \\
\text{real oriented Riemannian } n\text{-vector bundle} & \Longleftrightarrow & \text{principal } \mathrm{SO}(n)\text{-bundle} \\
\text{complex } n\text{-vector bundle} & \Longleftrightarrow & \text{principal } \mathrm{GL}_n(\mathbb{C})\text{-bundle} \\
\text{complex Hermitian } n\text{-vector bundle} & \Longleftrightarrow & \text{principal } \mathrm{U}(n)\text{-bundle}
\end{array}$$

Those equivalences will become much clearer in the context of holonomy, which is about to be discussed in the next chapter.

This idea of a twofold covering as in sec. 4.2 is now applied to bundles. For simplicity, only the signature $(n, 0)$ will be investigated in the following. Of course, the definitions also extend to semi-Riemannian vector bundles with signature (r, s). Let $E \xrightarrow{\pi} M$ be an oriented n-dimensional Riemannian vector bundle and $P_{\mathrm{SO}}(E) \xrightarrow{\tilde{\pi}} M$ the corresponding frame bundle with reduced structure group $\mathrm{SO}(n)$. The special orthogonal group has a 2-sheeted covering $\xi_0 : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$, which is induced by the twisted adjointment mapping for $n \geq 3$. Consider the following definition, which is dependent on the dimension n:

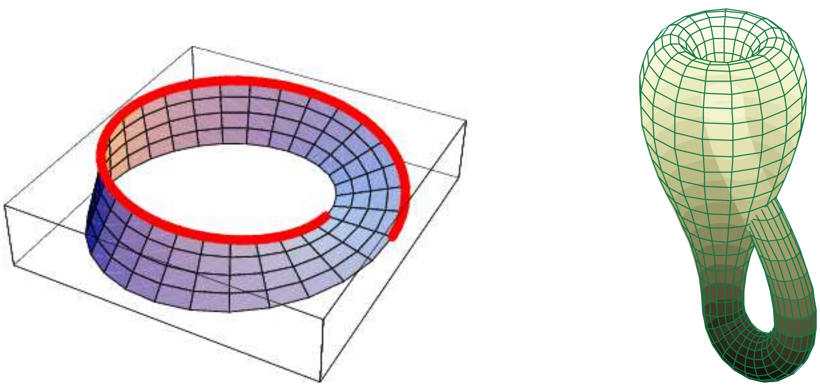
- $n \geq 3$: A **spin structure** $E \xrightarrow{\pi} M$ is a principal Spin(n)-bundle together with a 2-sheeted bundle covering map
$$\xi : P_{\mathrm{Spin}}(E) \twoheadrightarrow P_{\mathrm{SO}}(E)$$
such that Spin(n)-equivariance $\xi(pg) = \xi(p)\xi_0(g)$ holds for all $p \in P_{\mathrm{Spin}}(E)$ and all $g \in \mathrm{Spin}(n)$.
- $n = 2$: A spin structure is defined analogously with the group Spin(n) replaced by $\mathrm{Spin}(2) \cong \mathrm{U}(1) \cong \mathrm{SO}(2)$ and $\xi_0 : \mathrm{SO}(2) \twoheadrightarrow \mathrm{SO}(2)$ the connected 2-sheeted covering—one might imagine this covering as a circle $\mathrm{SO}(2) \approx S^1$ twisted once and projected onto SO(2).
- $n = 1$: Due to $\mathrm{SO}(1) = \{1\}$ and $\mathrm{Spin}(1) \cong \mathrm{O}(1) = \{\pm 1\}$, it follows $P_{\mathrm{SO}(E)} \cong M$ and a spin structure is simply defined to be a 2-sheeted covering of M.

Thus, by definition a spin structure on a real oriented Riemannian vector bundle is the natural extension of the twofold covering property of a single group to an entire bundle, as depicted in the following diagram with short exact sequences in the lines:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2 \arrow[r, hook] & \mathrm{Spin}(n) \arrow[rr, two heads, "\xi_0 = \tilde{\mathrm{Ad}}", "\text{2-sheeted covering}"'] \arrow[d, squiggly] & & \mathrm{SO}(n) \arrow[r] \arrow[d, squiggly] & 1 & \text{(single fiber)} \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r, hook] & P_{\mathrm{Spin}}(E) \arrow[rr, two heads, "\xi", "\text{spin structure}"'] \arrow[dr, "\text{bundle projection } \tilde{\pi}"'] & \circlearrowleft & P_{\mathrm{SO}}(E) \arrow[r] \arrow[dl, "\text{bundle projection } \pi"] & 1 & \text{(entire bundle)} \\
& & & M & & &
\end{tikzcd}$$

It remains to investigate the question of existence and uniqueness of spin structures. Similar to the case of orientability this question is answered by the global topology of the bundle and thus encoded in certain characteristic classes. Similar to the case of orientability (and as well discussed in [LM89, §2]) one can prove: There exists a spin structure on the vector bundle $E \xrightarrow{\pi} M$ if and only if the second Stiefel-Whitney class vanishes, i.e. $w_2(E) = 0 \in H^2(M; \mathbb{Z}_2)$. Furthermore, if $w_2(E) = 0$, then the distinct spin structures on E are in bijective correspondence with the elements of $H^1(M; \mathbb{Z}_2)$.

The two conditions $w_1(E) = 0$ (orientability) and $w_2(E) = 0$ (existence of spin structure) can be interpreted geometrically as follows: A vector bundle $E \xrightarrow{\pi} M$ is orientable if and only if the restriction of E to any 1-sphere embedded in the base space M yields a trivial product bundle. An oriented vector bundle (of dimension > 4) over a simply-connected base space is spin if and only if the restriction of E to any 2-sphere S^2 embedded in M is trivial.



(Möbius strip taken from MathWorld, Klein bottle taken from Wikipedia, authors unknown.)

FIGURE 4.1. The Möbius strip and the Klein bottle.

4.7. Clifford and spinor bundles

The existence of a spin structure on a given vector bundle allows to construct a spinor bundle, whose sections are what physicists call spinor fields. This is conceptually carried out in the same fashion as in sec. 4.5, i.e. by attaching a Clifford algebra representation as the fiber—using the associated bundle construction—and then restricting to the corresponding spin subgroup. Thus, in a certain sense, the rather technical details, which are about to follow, are most natural from the conceptual point of view.

Let $\rho_n : \mathrm{SO}(n) \longrightarrow \mathrm{SO}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ be the standard representation of the special orthogonal group. Since each orthogonal transformation of $\mathbb{R}^n$ induces an orthogonal transformation of $\mathrm{C}\ell(\mathbb{R}^n) = \mathrm{C}\ell_n$, the representation mapping ρ_n has a unique extension

$$\mathrm{c}\ell(\rho_n) : \mathrm{SO}(n) \longrightarrow \mathrm{Aut}\big(\mathrm{C}\ell(\mathbb{R}^n)\big)$$

to a representation onto the Clifford algebra. Let $E \xrightarrow{\pi} M$ be an oriented Riemannian vector bundle, i.e. a bundle of vector spaces with inner products. The corresponding **Clifford bundle** $\mathrm{C}\ell(E)$ is just the associated bundle of Clifford algebras determined by the inner products, i.e.

$$\mathrm{C}\ell(E) := P_{\mathrm{SO}}(E) \times_{\mathrm{c}\ell(\rho_n)} \mathrm{C}\ell(\mathbb{R}^n).$$

Since $\mathrm{C}\ell(E)$ is a bundle of algebras over M, the fiber-wise Clifford multiplication in $\mathrm{C}\ell(E)$ gives an algebra structure to the space of sections $\Gamma\big(\mathrm{C}\ell(E)\big)$.

Now the spin bundles encountered in physics can be defined. Let $E \xrightarrow{\pi} M$ be an oriented Riemannian vector bundle with a chosen spin structure $\xi : P_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$, which in particular requires $w_1(E) = w_2(E) = 0$. Let M be a left module over $\mathrm{C}\ell(\mathbb{R}^n)$, i.e. a space accompanied by a representation of $\mathrm{C}\ell(\mathbb{R}^n)$ on M which provides an action

$$\mathrm{C}\ell(\mathbb{R}^n) \times M \longrightarrow M.$$

Furthermore, let $\Delta : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(M)$ be the representation induced by restricting the $\mathrm{C}\ell(\mathbb{R}^n)$-action on M to $\mathrm{Spin}(n) \subset \mathrm{C}\ell^0(\mathbb{R}^n) \subset \mathrm{C}\ell(\mathbb{R}^n)$.

- *Real case:* A **real spinor bundle** to E is an associated bundle of the form
$$S(E) := P_{\mathrm{Spin}}(E) \times_\Delta M.$$
- *Complex case:* Similarly, a **complex spinor bundle**[d] to E is a bundle of the form
$$S_{\mathbb{C}}(E) := P_{\mathrm{Spin}}(E) \times_\Delta M_{\mathbb{C}}$$
where $M_{\mathbb{C}}$ is a complex left module over $\mathrm{C}\ell(\mathbb{R}^n) \otimes \mathbb{C} = \mathrm{C}\ell(\mathbb{C}^n) = \mathbb{C}\ell_n$.

The smooth sections of either bundle are called (real or complex) **spinors** or **spinor fields**. The $\mathbb{Z}_2$-graded eigenspace splitting $\mathrm{C}\ell(\mathbb{R}^n) = \mathrm{C}\ell^0(\mathbb{R}^n) \oplus \mathrm{C}\ell^1(\mathbb{R}^n)$ observed in sec. 4.4 for a single Clifford algebra $\mathrm{C}\ell(\mathbb{R}^n)$ as induced by the reflection mapping α carries on to the Clifford bundle, i.e. there is an analogous splitting

$$\mathrm{C}\ell(E) = \mathrm{C}\ell^0(E) \oplus \mathrm{C}\ell^1(E)$$

into the even and odd part of the Clifford bundle. Now the chirality splitting found in the real and complex spinor representations can be translated to the bundle formalism.

- *Real case:* Suppose $n = 4m+2$ and $S(E) \xrightarrow{\tilde\pi} M$ is the irreducible real spinor bundle associated to $E \xrightarrow{\pi} M$. Define a global section ω of $\mathrm{C}\ell(E)$ by setting $\omega(b) = e_1 \cdots e_n$ at $b \in M$ where $\{e_1, \dots, e_n\}$ is any positively oriented orthonormal basis of the fibre E_b. Again, $\omega^2 = 1$, which determines the eigenspace decomposition
$$S(E) = S^+(E) \oplus S^-(E).$$

[d]There is a similar notion, called a **spin$^{\mathbb{C}}$ structure**, which can only be used to construct complex spinor bundles. The corresponding existence conditions are less restrictive than for (real) spin structures, thus any bundle admitting a spin structure also admits a spin$^{\mathbb{C}}$ structure. In particular, the tangent bundle of any 4-dimensional (pseudo-)Riemannian manifold admits a spin$^{\mathbb{C}}$ structure. See [LM89, app. D] for further information.

- *Complex case:* Suppose $n = 2m$ is even and $S_{\mathbb{C}}(E) \xrightarrow{\tilde{\pi}} M$ is the irreducible complex spinor bundle to the vector bundle $E \xrightarrow{\pi} M$. Consider the global section $\omega_{\mathbb{C}}$ of $C\ell(E) \otimes \mathbb{C}$ which at a fixed point $b \in M$ is given by

$$\omega_{\mathbb{C}}(b) = \mathrm{i}^m e_1 \cdots e_{2m}$$

for any positively oriented orthonormal basis $\{e_1, \ldots, e_{2m}\}$ of E_b. Then the property $(\omega_{\mathbb{C}})^2 = 1$ holds. Just like in sec. 4.4 define $S^+_{\mathbb{C}}(E)$ and $S^-_{\mathbb{C}}(E)$ to be the ± 1 eigenbundles for Clifford multiplication by $\omega_{\mathbb{C}}$ to arrive at the chirality splitting

$$S_{\mathbb{C}}(E) = S^+_{\mathbb{C}}(E) \oplus S^-_{\mathbb{C}}(E).$$

In the case $n = 8m + 2$ there is the relationship $S^{\pm}(E) \otimes \mathbb{C} \cong S^{\pm}_{\mathbb{C}}(E)$ between the real and complex spinor representations. This corresponds to the fact, that in these dimensions $\Delta^{\pm}_{8m+2}$ are the complexifications of real representations.[e]

4.8. Usage in physics

To close this rather technical chapter, the introduced notions are brought into contact to the terms usually encountered in the physical literature, e.g. it is discussed what Majorana, Dirac or Weyl spinors are. First, recall how spinors arise in physics: The relativistic picture of physics relies on a d-dimensional space-time with 1 temporal and $d-1$ spatial degrees of freedom and described by $\mathbb{R}^d$ with quadratic form $q_{d-1,1}$ or $q_{1,d-1}$, i.e. under the quadratic form the temporal and spatial directions square differently. The naive invariance group preserving this quadratic form is either

$$\begin{aligned} \mathcal{L}_{1,d-1} &:= \mathrm{O}(\mathbb{R}^d, q_{1,d-1}) \cong \mathrm{O}(1, d-1) \quad \text{or} \\ \mathcal{L}_{d-1,1} &:= \mathrm{O}(\mathbb{R}^d, q_{d-1,1}) \cong \mathrm{O}(d-1, 1), \end{aligned}$$

which is called the **(full) Lorentz group** in either the "mostly plus" $(d-1,1)$ or "mostly minus" $(1, d-1)$ signature. The irreducible representations of the Lorentz group are conventionally called **tensor representations**.

However, due to the description of symmetries in terms of infinitesimal variations, the elements of the Lorentz group are usually described using the exponential mapping by the corresponding Lie algebra. Since the Lorentz group consists of four connection components, information is lost of the three connection components not containing the unit element, which are usually associated with time reversal (T), spatial reflection (P) or both (PT). Using the infinitesimal description by the Lorentz algebra, only the **restricted Lorentz group**

$$\begin{aligned} \mathcal{L}^{\uparrow}_{+,(1,d-1)} &:= \mathrm{SO}^0(\mathbb{R}^d, q_{1,d-1}) \cong \mathrm{SO}^0(1, d-1) \quad \text{or} \\ \mathcal{L}^{\uparrow}_{+,(d-1,1)} &:= \mathrm{SO}^0(\mathbb{R}^d, q_{d-1,1}) \cong \mathrm{SO}^0(d-1, 1) \end{aligned}$$

is retrieved, which is the unit element's connection component of the full Lorentz group. Since it does not contain time reversal or space reflection, it is usually called the proper, orthochronous part of the Lorentz group. Furthermore, there are isomorphisms

$$\mathrm{Lie}\,\mathcal{L}_{1,d-1} = \mathfrak{o}(1, d-1) \cong \mathfrak{so}(1, d-1) \cong \mathfrak{spin}(1, d-1) \cong \mathfrak{pin}(1, d-1) \cong \mathrm{Lie}\,\mathrm{Pin}(1, d-1)$$

and analogous for the $(d-1, 1)$ signature, such that the infinitesimal description of the Lorentz group cannot be distinguished from the description of its universal covering group. Thus, one

[e]In the context of superstring theory this result is in strong favor of the 10-dimensional space-time of the supercritical superstring theories, since 10 is the smallest possible dimension, where a 2-dimensional surface (string worldsheet) can be embedded in a non-trivial way such that both admit the same type of spinors.

d	**W**	**M**	**MW**	**SM**	**pM**	**pMW**	**pSM**
1					✓		
2	✓	✓	✓		✓	✓	
3		✓					
4	✓	✓					✓
5							✓
6	✓			✓			✓
7				✓			
8	✓			✓	✓		
9					✓		
10	✓	✓	✓		✓	✓	
11		✓					

TABLE 4.4. (S)pinor representations with respect to the "mostly plus" spacetime signature in d dimensions. W are complex Weyl spinors, M are Majorana pinors, MW are Majorana-Weyl spinors, SM are symplectic Majorana pinors, pM are pseudo-Majorana pinors, pMW are pseudo Majorana-Weyl spinors and pSM are pseudo-symplectic Majorana pinors.

recovers tensor and spinor representations, which stem from the respective covering groups

$$\begin{tikzcd}
\tilde{\mathcal{L}} := \mathrm{Pin}(1,d-1) \arrow[r, two heads, "2:1"] & \mathrm{O}(1,d-1) = \mathcal{L} & \text{(full Lorentz group)} \\
\tilde{\mathcal{L}}_+ := \mathrm{Spin}(1,d-1) \arrow[r, two heads, "2:1"] \arrow[u, hook, "\text{subgroup}"'] & \mathrm{SO}(1,d-1) = \mathcal{L}_+ \arrow[u, hook, "\text{subgroup}"] & \text{(proper Lorentz group)} \\
\tilde{\mathcal{L}}_+^\uparrow := \mathrm{Spin}^0(1,d-1) \arrow[r, two heads, "2:1"] \arrow[u, hook, "\text{subgroup}"'] & \mathrm{SO}^0(1,d-1) = \mathcal{L}_+^\uparrow \arrow[u, hook, "\text{subgroup}"] & \text{(restricted Lorentz group)}
\end{tikzcd}$$

To summarize: $\mathcal{L}$ is the full Lorentz group with four connection components, $\mathcal{L}_+$ consists of two connection components at which the chirality splitting occurs in the representation theory of the covering groups, and $\mathcal{L}_+^\uparrow$ is the sole connection component of the unit element, which is fully described by the Lorentz algebra. The covering groups will be denoted with a tilde to simplify notation. In particular, the pinor representations are representations of $\tilde{\mathcal{L}}$, which might split in either one or two spinor representations of $\tilde{\mathcal{L}}_+$. This issue is completely neglected in the physical terminology, such that in physics everything is called a "spinor" even if it is in fact a pinor representation. The following notions occur in physics:

- A real representation is called **Majorana**, which gives rise to Majorana pinors and Majorana spinors in certain dimensions, albeit the former is never emphazised in the physical literature. In the physical context, Majorana fermions describe particles which are their own antiparticles.
- Likewise, quaternionic representations are called **symplectic Majorana**, which is very seldom used in physics.
- In those cases, where a real pinor representation is only available for the other possible signature, physicists introduce **pseudo-Majorana spinors**, i.e. a "mostly minus" pseudo-Majorana spinor refers to the pinor representation in the "mostly plus" signature and vice versa. However, this is a concept without any mathematical justification and should be avoided.
- Similarly, there are **pseudo-symplectic Majorana spinors** introduced in the same fashion.

- In certain dimensions a chirality splitting $\Delta = \Delta^+ \oplus \Delta^-$ of a pinor representations occurs, as was explained in sec. 4.5. The inequivalent irreducible spinor representations Δ^+ and Δ^- are called **Weyl spinors** of the respective chirality.
- If both a chirality splitting occurs and the spinor representations are real, they are called **Majorana-Weyl spinors**. This only happens in dimensions $d \equiv 2 \bmod 8$, which gives quite a distinct appeal to 2d worldsheets moving in a 10d space-time.
- The famous **Dirac spinor** is usually identified with a complex pinor representation in a dimension admitting a chirality splitting in two Weyl spinors. However, sometimes it refers just to a general pinor representation of the full Lorentz group.

Consider the special case of a 4-dimensional space-time. As mentioned, physicists usually use the complex spin representations, which are completely independent of the choice of signature. Thus, $C\ell_{1,3} \otimes_{\mathbb{R}} \mathbb{C} \cong C\ell_{3,1} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\ell_4 \cong \mathbb{C}(4)$ has a complex 4-dimensional representation—the familiar 4d Dirac spinor (or rather pinor)—which admits a chirality splitting $\Delta_4^{\mathbb{C}} \cong \Delta_4^{\mathbb{C}+} \oplus \Delta_4^{\mathbb{C}-}$ into two inequivalent 4d Weyl spinors. In fact, this suffices for almost everything in physics. In the "mostly minus" signature the corresponding real Clifford algebra $C\ell_{1,3} \cong \mathbb{H}(2)$ gives rise to an quaternionic representation, which can also be regarded as the Dirac spinor via $\mathbb{H}(2) \subset \mathbb{C}(4)$. However, the quaternionic structure is never emphazised in physics. Finally, the Clifford algebra $C\ell_{3,1} \cong \mathbb{R}(4)$ gives rise to a Majorana pinor representation in the "mostly plus" signature.

CHAPTER 5

Riemannian Geometry and Holonomy

In this chapter the theory of connections introduced in the framework of gauge theory will be specialized to the case of Riemannian geometry, focusing in particular on the issues of Riemannian curvature and holonomy groups. Furthermore, the general Dirac operator will be introduced, which naturally links Riemannian geometry to the spinor concepts introduced in the last chapter. The exposition follows the respective chapters of [Joy00], whereas the book [dC92] provides a very detailed general introduction into the subject of Riemannian geometry.

5.1. Connections, curvature and parallel transport in vector bundles

In the context of principal bundles, the curvature $\Omega \in \Omega^2_P(\mathfrak{g})$ was introduced as the covariant exterior derivative of the connection form $\omega \in \Omega^1_P(\mathfrak{g})$. The central idea of a connection was to provide an external choice of horizontal complements to the vertical subspaces induced by the principal bundle projection. Via the concept of associated bundles the covariant derivative of vector bundles was introduced. Usually, Riemannian geometry starts right here at this point, where the derived covariant derivative is used as a fundamental definition: Let $E \xrightarrow{\pi} M$ be a vector bundle on the manifold M. A **(affine) connection** on E is a linear map $\nabla : \Gamma(E) \longrightarrow \Gamma(\mathrm{T}^*M \otimes E)$ such that the generalized Leibniz rule

$$\nabla(f\sigma) = f\nabla\sigma + \mathrm{d}f \otimes \sigma$$

is satisfied whenever $\sigma \in \Gamma(E)$ is a smooth section of the vector bundle and $f : M \longrightarrow \mathbb{R}$ a real-valued smooth function on M.

Essentially, the quite lengthy process to derive the resulting form of the covariant derivative within the vector bundle from the choice of a connection on the frame bundle, using connection forms, exterior covariant derivatives and associated bundles, is truncated and the resulting object is directly defined to have the necessary properties. Thus, the choice of a connection in the "covariant derivative sense" used here is a special case of the choice of a horizontal subbundle as used in the gauge theory chapter.

As mentioned before, the mapping $\nabla_V : \Gamma(E) \longrightarrow \Gamma(E)$ arises from the contraction of ∇ with a smooth vector field $V \in \mathfrak{X}(M)$ and can be interpreted as the directional (covariant) derivative on the vector bundle $E \xrightarrow{\pi} M$. In particular, the Leibniz rule in this case reads

$$\nabla_{\alpha V}(f\sigma) = \alpha f \nabla_V \sigma + \alpha(Vf)\sigma,$$

where $\alpha : M \longrightarrow \mathbb{R}$ is another real-valued function and Vf the directional derivative of f.

The curvature of a connection in the "distributional sense" is a 2-form $\Omega \in \Omega^2_P(\mathfrak{g})$ as introduced in the chap. 2. The same information was shown to be encapsulated in the gauge-field strength $F \in \Omega^2_M(\operatorname{ad} P)$, which is in fact a section of the bundle $\Lambda^2\mathrm{T}^*M \otimes \operatorname{ad} P$. Due to

$$\mathfrak{gl}_n(\mathbb{R}) \cong \operatorname{End}(\mathbb{R}^n)$$

it follows for the principal $\mathrm{GL}_n(\mathbb{R})$-frame bundle $\operatorname{Fr}(E) \xrightarrow{\tilde{\pi}} M$ associated to $E \xrightarrow{\pi} M$, that the adjoint bundle is $\operatorname{ad}\operatorname{Fr}(E) \cong \operatorname{End}(E)$, which is the bundle of endomorphisms of the fibers of the vector bundle $E \xrightarrow{\pi} M$.

Thus, the **curvature** of an affine connection—in the vector bundle sense used here—is a section $R_\nabla \in \Gamma\big(\Lambda^2 \mathrm{T}^* M \otimes \mathrm{End}(E)\big) = \Omega^2_M\big(\mathrm{End}(E)\big)$ given by[a]

$$\begin{aligned} R_\nabla(V,W)\sigma &= \nabla_V \nabla_W \sigma - \nabla_W \nabla_V \sigma - \nabla_{[V,W]}\sigma \\ &= \big([\nabla_V, \nabla_W] - \nabla_{[V,W]}\big)\sigma. \end{aligned}$$

Once again, this is a result derived from the more general principal bundle approach. Obviously, the geometric interpretation in this case is that curvature measures the failure of the directional (covariant) derivative to form a Lie algebra. This is also the point of view in which physicists regard curvature. A connection with vanishing curvature is simple called **flat**.

A connection ∇ on the tangent bundle $\mathrm{T}M \xrightarrow{\pi} M$ of a smooth manifold is called the **connection of the manifold**. The **torsion** of such a connection ∇ is an important property, especially in the prospect of Riemannian geometry. It is described by the unique, smooth section $T_\nabla \in \Gamma(\Lambda^2 \mathrm{T}^* M \otimes \mathrm{T}M) = \Omega^2_M(\mathrm{T}M)$ that satisfies

$$T_\nabla(V,W) = \nabla_V W - \nabla_W V - [V,W]$$

for each vector field $V, W \in \Gamma(TM) = \mathfrak{X}(M)$. A connection ∇ on $\mathrm{T}M$ with $T_\nabla = 0$ is called **torsion-free**—this will be a central requirement in Riemannian geometry.

Let $E \xrightarrow{\pi} M$ be a vector bundle over M with a connection ∇. Suppose $\gamma : [0,1] \longrightarrow M$ is a smooth curve between $\gamma(0) = p$ and $\gamma(1) = q$. Then for each vector $v \in E_p$ there exists a unique smooth section $\sigma \in \gamma^* E$ satisfying $\nabla_{\dot\gamma(t)}\sigma(t) = 0$ for all $t \in [0,1]$ with $\sigma(0) = v$, where $\gamma^* E$ is the pullback bundle induced by the curve and $\dot\gamma(t)$ the vector field induced by the "velocity" of the curve. This defines the **parallel transport mapping**

$$\begin{aligned} P_\gamma : E_p &\xrightarrow{\cong} E_q \\ v &\mapsto \sigma(1) \end{aligned}$$

which in fact establishes an isomorphism between the fibers at the start and end point of the curve. The construction is easily generalized to piecewise smooth paths in the base space. If $\gamma_1\gamma_2$ denotes the linking of two curves with $\gamma_1(1) = \gamma_2(0)$, then $P_{\gamma_1\gamma_2} = P_{\gamma_2} \circ P_{\gamma_1}$ holds. Thus, besides the means of a derivative of vector bundle sections, a connection provides a natural prescription to move vectors in the bundle along a curve in the base space. Actually, the notion of a "connection" comes from this particular property, as a connection provides the means to "connect" (or rather relate) the vectors of different fibers of a vector bundle.

5.2. Holonomy

Now consider a closed curve $\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = \gamma(1) = p$ in the base space, which is called a **loop based at *p***. Thus, the parallel transport mapping $P_\gamma : E_p \xrightarrow{\cong} E_p$ induces an automorphism of the fiber E_p. This defines the ***p*-based holonomy group** $\mathrm{Hol}_p(\nabla)$ of the connection ∇ to be the automorphism subset

$$\mathrm{Hol}_p(\nabla) := \big\{P_\gamma \text{ for all loops } \gamma \text{ based at } p\big\} \subset \mathrm{Aut}(E_p) \cong \mathrm{GL}(E_p).$$

As the name suggests, $\mathrm{Hol}_p(\nabla)$ is not just a subset of $\mathrm{GL}(E_p)$ but a subgroup: Consider loops based at p, then the gluing of these loops provides a composition operation and curves parameterized in the opposite direction provide inverse elements, thus making the group structure manifest.

However, the really important point of this construction is its essential invariance from the choice of the base point: Let the points $p, q \in M$ be joined by a smooth curve $\gamma : [0,1] \longrightarrow M$, such that $\gamma(0) = p$ and $\gamma(1) = q$, and τ be any loop based at p. Then $\gamma\tau\gamma^{-1}$ is a loop based at q. Due to $P_{\gamma\tau\gamma^{-1}} = P_\gamma \circ P_\tau \circ P_\gamma^{-1}$ for each $P_\tau \in \mathrm{Hol}_p(\nabla)$, it follows that

$$P_\gamma \circ P_\tau \circ P_\gamma^{-1} \in \mathrm{Hol}_q(\nabla).$$

[a] The reader should be aware of the numerous sign conventions used for the curvature. The very popular textbook [dC92], for example, chooses a negative sign for R_∇.

After a choice of basis for E_p and E_q, respectively, the holonomy groups $\mathrm{Hol}_p(\nabla)$ and $\mathrm{Hol}_q(\nabla)$ can be considered as subgroups $H_p, H_q \subset \mathrm{GL}_k(\mathbb{R})$. If $g \in \mathrm{GL}_k(\mathbb{R})$ represents the holonomy automorphism P_γ, then it follows $gH_pg^{-1} = H_q$, i.e. the holonomy group is a subgroup of $\mathrm{GL}_k(\mathbb{R})$, defined up to conjugation. Thus, from now on, the specification of the base point p in $\mathrm{Hol}_p(\nabla)$ will be dropped.

It follows that the holonomy group is a global invariant of the chosen connection. Furthermore, for a simply-connected base space M the holonomy group $\mathrm{Hol}(\nabla)$ is a connected Lie group, see [Joy00, thm. 2.2.4]. When M is not simply-connected, it is convenient to consider a certain restriction of the holonomy group. A loop γ based at the point p is called **null-homotopic** if it can be deformed to a constant loop at p, i.e. the loop can degenerate to the mapping $\gamma(t) = p$. This defines the ***p*-based restricted holonomy group**

$$\mathrm{Hol}_p^0(\nabla) := \big\{P_\gamma \text{ for all null-homotopic loops } \gamma \text{ based at } p\big\} \subset \mathrm{Hol}_p(\nabla).$$

One proves independence of the base point p just like for the ordinary holonomy group, i.e. $\mathrm{Hol}^0(\nabla)$ is defined up to conjugation. If M is simply-connected, then $\mathrm{Hol}(\nabla) = \mathrm{Hol}^0(\nabla)$. Holonomy and restricted holonomy can be defined analogous for principal bundles, which is discussed in considerable detail in [Joy00, §2.3]. The general concept of holonomy will become important in sec. 8.6 and subsequent sections in the context of Calabi-Yau compactification.

5.3. Riemannian manifolds

An **Riemannian** or **Euclidean structure** on the real vector bundle $E \xrightarrow{\pi} M$ is a smooth section $g \in \Gamma(\mathrm{S}^2 E^*)$, which defines an positive inner product in every fiber E_p. Let $\sigma, \tau \in \Gamma(E)$ be two smooth sections. This was already introduced in sec. 4.6 in order to extend the idea of Clifford algebras and spinors to bundles. A connection ∇ on $E \xrightarrow{\pi} M$ is called **orthogonal** if for any vector field $V \in \mathfrak{X}(M)$ the identity

$$V\langle\sigma,\tau\rangle_g = \big\langle\nabla_V\sigma,\tau\big\rangle_g + \big\langle\sigma,\nabla_V\tau\big\rangle_g$$

holds. A **Riemannian metric** g for the smooth manifold M is a Riemannian structure on the tangent bundle $\mathrm{T}M$, i.e. it provides an inner product $g_p : \mathrm{T}_pM \times \mathrm{T}_pM \longrightarrow \mathbb{R}$ for each tangent space in a manner which varies smoothly from point to point. The metric can be understood as a specific section

$$g \in \Gamma\big(\mathrm{S}^2(\mathrm{T}^*M)\big)$$

of the symmetric tensor product of the cotangent bundle. Such a pair (M, g) of a smooth manifold and a metric is called a **Riemannian manifold**. Let $\gamma : [0,1] \longrightarrow M$ be a smooth curve within a Riemannian manifold, then $\gamma'(t) \in \mathrm{T}_{\gamma(t)}M$ and the **length of the curve** with respect to the Riemannian metric is defined by

$$L(\gamma) := \int_0^1 \|\gamma'(t)\|_g \,\mathrm{d}t,$$

where the norm is point-wise induced via $\|v\|_{g_p}^2 = \langle v, v\rangle_{g_p}$. The generalization for piecewise smooth curves is obvious. For two points $p, q \in M$ let $\Omega(p,q)$ denote the set of all piecewise smooth paths from p to q, i.e. mappings $\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = p$ and $\gamma(1) = q$. A distance function—in the sense of a metric as introduced in the first chapter—is then given by the infimum of all the lengths $\{L(\gamma) : \gamma \in \Omega(p,q)\}$. This provides a sense of distance on a Riemannian manifold and particularly turns it into a metric space. Thus, Riemannian manifolds have canonical topologies introduced via unions of open balls. An **isometry** of M is a distance-preserving diffeomorphism $M \xrightarrow{\approx} M$.

Given a Riemannian manifold (M, g), the "fundamental theorem of Riemannian geometry" provides the existence of a unique, torsion-free, orthogonal connection ∇ on the tangent bundle $\mathrm{T}M \xrightarrow{\pi} M$ with $\nabla g = 0$, called the **Levi-Civita connection**. In particular, the inner product of the covariant derivative and a vector field can be expressed as follows:

$$\begin{aligned}\left\langle \nabla_Y X, Z\right\rangle_g = \frac{1}{2}\Big(& X\langle Y, Z\rangle_g + Y\langle Z, X\rangle_g - Z\langle X, Y\rangle_g \\ & - \langle X, [Y, Z]\rangle_g - \langle Y, [X, Z]\rangle_g - \langle Z, [X, Y]\rangle_g\Big).\end{aligned} \tag{5.1}$$

Furthermore, any connection on the tangent bundle $\mathrm{T}M \xrightarrow{\pi} M$—in particular the Levi-Civita connection—canonically extends to a connection on $\mathrm{T}M^{\otimes k} \otimes \mathrm{T}^*M^{\otimes l} \xrightarrow{\tilde{\pi}} M$, i.e. on a Riemannian manifold all tensor power bundles have a distinguished connection induced by the metric g.

Note that the inner product in a Riemannian manifold is positive definite. This condition is relaxed by the introduction of **pseudo-Riemannian manifolds** and the fundamental theorem also holds for such manifolds. The particular class of n-dimensional pseudo-Riemannian manifolds with metric signature $(1, n-1)$ or $(n-1, 1)$ are called **Lorentz manifolds** in obvious resemblance to the associated physical framework.[b] The **Riemannian curvature** R of the metric g is defined by the curvature R_∇ of the Levi-Civita connection ∇ on the tangent bundle $\mathrm{T}M$. The Bianchi identity $\mathrm{d}^\#\Omega = 0$ translates to

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$

for vector fields $X, Y, Z \in \mathfrak{X}(M)$. Furthermore, $\langle R(X, Y)Z, W\rangle_g = \langle R(Z, W)X, Y\rangle_g$ holds for any vector fields $W, X, Y, Z \in \mathfrak{X}(M)$.

5.4. Index notation and Riemannian curvature

In this section most of the material will be translated to the coordinate-dependent index notation commonly used in physics. Let $(x_1, \ldots, x_n)$ be local coordinates on $U \subset M$, then the metric g takes the form

$$g|_U = \sum_{i,j=1}^n g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j,$$

where $g_{ij} : U \longrightarrow \mathbb{R}$ are the component functions of the **metric tensor**. If $\partial_i := \frac{\partial}{\partial x^i}$ denotes the basis vectors of the tangent spaces as induced by the local coordinates, the metric tensor is given by $g_{ij} = \langle \partial_i, \partial_j\rangle_g$, i.e.

$$g_{ij}(p) = \left\langle \left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \right\rangle_{g_p}.$$

Due to linearity, a connection is completely determined by its effect on the basis elements, thus formally the covariant derivative can be expressed by

$$\nabla_{\partial_j}(\partial_k) = \sum_{i=1}^n \Gamma^i_{jk}\partial_i, \tag{5.2}$$

where $\Gamma^i_{jk} : U \longrightarrow \mathbb{R}$ are the **Christoffel symbols** with respect to the chosen local coordinates. To give a reasonable formula for the Christoffel symbols, consider

$$\begin{aligned}&\left\langle \nabla_{\partial_j}(\partial_k), \partial_i\right\rangle_g = \sum_{l=1}^n \left\langle \Gamma^l_{jk}\partial_l, \partial_i\right\rangle_g = \sum_{l=1}^n \Gamma^l_{jk} g_{li} \\ \overset{(5.1)}{\iff} &\sum_{l=1}^n \Gamma^l_{jk} g_{li} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right).\end{aligned}$$

[b] Recall that in the following chapters the "mostly plus" signature $(-{+}{+}\ldots{+})$ will be used.

Since $g = (g_{ij})$ is a non-degenerate symmetric matrix, it is invertible. The point-wise inverse matrix for g will be denoted $g^{-1} = (g^{ij})$, and thus the equation

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^{n} g^{il}\left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right) = \frac{1}{2}\sum_{l=1}^{n} g^{il}\left(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}\right)$$

provides an easy way to calculate the local behavior of the Levi-Civita connection in terms of the Christoffel symbols, as used in general relativity, for example.

Likewise, the Riemannian curvature $R = R_\nabla \in \Gamma\big(\Lambda^2 \mathrm{T}^*M \otimes \mathrm{End}(\mathrm{T}M)\big)$ of the Levi-Civita connection can be expressed as components ${R^a}_{bcd}$ due to $\mathrm{End}(\mathrm{T}M) = \mathrm{T}^*M \otimes \mathrm{T}M$. Any metric g induces a (point-wise) canonical isomorphism $\mathrm{T}M \cong \mathrm{T}^*M$ via

$$\begin{aligned} \mathrm{T}_pM &\xrightarrow{\cong} \mathrm{T}^*_pM \\ v &\mapsto \langle v, . \rangle_{g_p}, \end{aligned}$$

that effectively raises and lowers the components indices in the physical notation. Thus, using the isomorphism $\mathrm{T}M \otimes \Lambda^2\mathrm{T}^*M \otimes \mathrm{T}^*M \cong \mathrm{T}^*M^{\otimes 2} \otimes \Lambda^2 \mathrm{T}^*M$ induced by the metric, one defines $R_{abcd} = g_{ae}{R^e}_{bcd}$, which is subject to the symmetries

$$\begin{aligned} &R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab}, && \text{(index symmetries)} \\ &R_{abcd} + R_{adbc} + R_{acdb} = 0, && \text{(first Bianchi identity)} \\ &\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0, && \text{(second Bianchi identity)} \end{aligned}$$

and usually called the **Riemann curvature tensor**. To give an explicit expression of the curvature tensor in terms of local components, first note the vanishing $[\partial_i, \partial_j] = 0$ of the vector field bracket for the local vector fields induced by the local coordinates $(x_1, \ldots, x_n)$. Thus, using the local behavior (5.2) of the Levi-Civita connection in terms of Christoffel symbols and the generalized Leibniz rule, it follows

$$\begin{aligned} R(\partial_i, \partial_j)\partial_k &= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k \\ &= \nabla_{\partial_i}\left(\sum_{l=1}^{n}\Gamma^l_{jk}\partial_l\right) - \nabla_{\partial_j}\left(\sum_{l=1}^{n}\Gamma^l_{ik}\partial_l\right) \\ &= \sum_{l=1}^{n}\Big[\Gamma^l_{jk}\left(\nabla_{\partial_i}\partial_l\right) + \left(\partial_i\Gamma^l_{jk}\right)\partial_l - \Gamma^l_{ik}\left(\nabla_{\partial_j}\partial_l\right) - \left(\partial_j\Gamma^l_{ik}\right)\partial_l\Big] \\ &= \sum_{l=1}^{n}\left[\sum_{r=1}^{n}\left(\Gamma^r_{jk}\Gamma^l_{ir} - \Gamma^r_{ik}\Gamma^l_{jr}\right) + \frac{\partial\Gamma^l_{jk}}{\partial x^i} - \frac{\partial\Gamma^l_{ik}}{\partial x^j}\right]\partial_l \\ &= {R^l}_{ijk}\partial_l \\ \iff\ {R^l}_{ijk} &= \Gamma^l_{ir}\Gamma^r_{jk} - \Gamma^l_{jr}\Gamma^r_{ik} + \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} \end{aligned}$$

for the curvature tensor. This is the definition that can be found in any textbook on general relativity. Using the explicit formula for the Christoffel symbols, it can be expressed in a (quite lengthy) sum of partial derivatives of the metric components.

A particular important type of curvature, is the **Ricci curvature** defined by the average sectional curvature, see [dC92, §4.4] for details,

$$\mathrm{Ric}_p(v) := \frac{1}{n-1}\sum_{i=1}^{n-1}\big\langle R(v, w_i)v, w_i\big\rangle_{g_p} = \frac{1}{n-1}R_{rs}v^r v^s,$$

where $w_i \in \mathrm{T}_pM$ are orthonormal basis vectors of the hyperplane in T_pM orthogonal to $v = \sum_{r=1}^{n} v^r\partial_r \in \mathrm{T}_pM$. Using the local coordinate representation of the Riemann curvature tensor, one quickly derives $R_{rs} = {R^p}_{rps}$, where the double p refers to the trace over the respective indices. Furthermore, there is the **scalar Ricci curvature**

$$K_p := \frac{1}{n}\sum_{j=1}^{n} \mathrm{Ric}_p(w_j) = \frac{1}{n(n-1)}\left\langle R(w_i, w_j)w_i, w_j\right\rangle_{g_p}$$

which is the average of the Ricci curvature over all the directions of an orthonormal basis of the tangent space T_pM. It can be expressed as the trace $K = g^{rs}R_{rs}$ of the Ricci curvature tensor.[c] Ricci curvature is a measure of the volume distortion in a curved environment, i.e. it encodes the difference of an n-dimension volume of a Riemannian manifold with the comparable volume in Euclidean $\mathbb{R}^n$. In particular, Ricci curvature is most important in general relativity, where the vacuum is (without a cosmological constant) described as a Ricci-flat Lorentz-manifold.

There is an important theorem in (pseudo-)Riemannian geometry which relates the scalar curvature of a 2-dimensional manifold directly to its topology. More precisely, let M be a compact, 2-dimensional (pseudo-)Riemannian manifold without boundary[d] and K being the Ricci scalar curvature, then the relation

$$\int_M R\,\mathrm{d}A = -4\pi\chi(M)$$

is called the **Gauß-Bonnet theorem**. This is important in string perturbation theory, as it determines the powers of the couplings via the right-hand side.

5.5. Riemannian holonomy groups

In sec. 5.2 holonomy groups and restricted holonomy groups (for non-simply-connected manifolds) were defined with respect to arbitrary connections in vector bundles. Using the distinguished Levi-Civita connection of a (pseudo-)Riemannian manifold, this gives rise to holonomy groups directly depending on the manifolds geometry.

The metric g is a covariantly constant tensor on any n-dimensional Riemannian manifold M with Levi-Civita connection ∇. Since the metric g_p is preserved in parallel transport of any vector $v \in \mathrm{T}_pM$ along loops based at $p \in M$, the **Riemannian holonomy group** $\mathrm{Hol}(g)$, i.e. the holonomy group $\mathrm{Hol}(\nabla)$ of $\mathrm{T}M$ with respect to the Levi-Civita connection, must be a subgroup of $\mathrm{O}(n) \subset \mathrm{GL}_n(\mathbb{R})$. Similarly, the **restricted Riemannian holonomy group** $\mathrm{Hol}^0(g)$ is defined to be $\mathrm{Hol}^0(\nabla)$ with ∇ being the Levi-Civita connection.

Naturally, the question arises, which subgroups of $\mathrm{SO}(n)$ can appear as actual Riemannian holonomy groups for a given manifold M. To give an appropriate answer to this important question, certain technicalities have to be discussed first.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds. The tangent space of the cartesian product manifold $M_1 \times M_2$ splits according to

$$\mathrm{T}_{(p_1,p_2)}(M_1 \times M_2) \cong \mathrm{T}_{p_1}M_1 \times \mathrm{T}_{p_2}M_2.$$

Thus, there is a natural induced **product metric** $(g_1 \times g_2)|_{(p_1,p_2)} := g_1|_{p_1} + g_2|_{p_2}$, such that $(M_1 \times M_2, g_1 \times g_2)$ is the **Riemannian product** of Riemannian manifolds. The induced Riemannian holonomy group of such a product is

$$\mathrm{Hol}(g_1 \times g_2) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2).$$

Conversely, this construction gives rise to the notion of reducibility of manifolds: A Riemannian manifold is called **reducible** if it is isomorphic to a Riemannian product manifold. As a refinement, (M, g) is called **locally reducible** if every point has a reducible open neighborhood. Then (M, g) is called an **irreducible manifold** if it is not locally reducible.

A function is called **involutive** if it is its own inverse, i.e. $f \circ f = \mathrm{Id}$. In particular, involutive isometries are of special interest, e.g. the mapping of reflection at an arbitrary line in Euclidean space is an involutive isometry, same as a 180 degree rotation around any

[c] In the physical literature the Ricci scalar curvature is usually denoted as R. However, this would lead to confusion with the coordinate-independent Riemann curvature tensor $R = R_\nabla$.

[d] There is a generalization of the Gauß-Bonnet theorem to manifolds with boundary.

	SO(n)-holonomy	Kähler	Calabi-Yau	Hyper-kähler	quat. Kähler	G_2-holonomy	Spin(7)-holonomy
$n = 1, 2, 3$:	$\checkmark$						
$n = 4$:	$\checkmark$	$\checkmark$	$\checkmark(2)$				
$n = 5$:	$\checkmark$						
$n = 6$:	$\checkmark$	$\checkmark$	$\checkmark(2)$				
$n = 7$:	$\checkmark$					$\checkmark(1)$	
$n = 8$:	$\checkmark$	$\checkmark$	$\checkmark(2)$	$\checkmark(3)$	$\checkmark$		$\checkmark(1)$
$n = 4m + 1$:	$\checkmark$						
$n = 4m + 2$:	$\checkmark$	$\checkmark$	$\checkmark(2)$				for
$n = 4m + 3$:	$\checkmark$						$m \geq 2$
$n = 4m + 4$:	$\checkmark$	$\checkmark$	$\checkmark(2)$	$\checkmark(m + 1)$	$\checkmark$		

TABLE 5.1. Existence of special Riemannian manifolds in dimension n. The numbers put in parentheses denote the dimension of the space of parallel spinors for the respective type of manifold, see sec. 5.8.

axis. A Riemannian manifold (M, g) is **locally symmetric** if any point $p \in M$ has a open neighborhood $U_p \subset M$ and an involutive isometry $s_p : U_p \longrightarrow U_p$ with a single, unique fixed point p, i.e. $s_p(p) = p$. Likewise, (M, g) is **(globally) symmetric** if $U_p = M$ for all $p \in M$. The manifold (M, g) is called **non-symmetric**, if it is not locally symmetric. By virtue of the local version of the theorem of Cartan-Ambrose-Hicks (cf. [Joy00, thm. 3.3.8]), M is locally symmetric if and only if its curvature tensor is covariantly constant, i.e. $\nabla R = 0$.

In 1955, Berger proved an important classification result, cf. [Joy00, §3.4]: Suppose M is a simply-connected, irreducible, non-symmetric, n-dimensional Riemannian manifold (M, g), then exactly one of the following seven cases holds:

(1) $\mathrm{Hol}(g) = \mathrm{SO}(n)$: Those are (ordinary) Riemannian manifolds. The holonomy group $\mathrm{SO}(n)$ provides that every tangent vector $v \in \mathrm{T}_pM$ is transformed by a simple rotation when parallel transported along a loop based at p.
(2) $\mathrm{Hol}(g) = \mathrm{U}(m) \subset \mathrm{SO}(2m)$ for $n = 2m \geq 4$: Riemannian manifolds with $\mathrm{U}(m)$-holonomy are called **Kähler manifolds**. This is equivalent to g being a specific Hermitean metric (see next section), which is the canonical complex equivalent to a Riemannian metric.
(3) $\mathrm{Hol}(g) = \mathrm{SU}(m) \subset \mathrm{SO}(2m)$ for $n = 2m \geq 4$: $\mathrm{SU}(m)$-holonomy manifolds are called **Calabi-Yau manifolds**, and in particular are locally Ricci-flat Kähler manifolds.
(4) $\mathrm{Hol}(g) = \mathrm{Sp}(m) \subset \mathrm{SO}(4m)$ for $n = 4m \geq 8$: The **Hyperkähler manifolds** are special types of Calabi-Yau and Kähler manifolds, as $\mathrm{Sp}(m) \subseteq \mathrm{SU}(2m) \subset \mathrm{U}(2m)$, i.e. they are locally Ricci-flat Kähler manifolds.
(5) $\mathrm{Hol}(g) = \mathrm{Sp}(m) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4m)$ for $n = 4m \geq 8$: Riemannian metrics g with $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$-holonomy are called **quaternionic Kähler**, they are Einstein manifolds but not Ricci-flat.
(6) $\mathrm{Hol}(g) = G_2 \subset \mathrm{SO}(7)$ for $n = 7$: Having just 14 dimensions, G_2 is the smallest of the five exceptional Lie groups that appear in Cartan's classification of simple Lie algebras. These manifolds are sometimes called **Joyce manifolds**. G_2-holonomy manifolds have gained much interest in the context of M-theory compactification.
(7) $\mathrm{Hol}(g) = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$ for $n = 8$: The group Spin(7) is the unique twofold universal covering group of SO(7), as discussed in the last chapter. First examples of Spin(7)-manifolds were constructed by Joyce, they are of some interest in (unrealistic) compactifications of M-theory to 2+1 dimensions.

In tab. 5.1 the possible types of simply-connected, irreducible, non-symmetric Riemannian manifolds of n dimensions are indicated. The appearance of G_2- and Spin(7)-manifolds in dimensions $n = 7$ and 8 are quite striking. The different types of special holonomy manifolds can be grouped together as follows:

- **Kähler holonomy groups** are $\mathrm{U}(m)$, $\mathrm{SU}(m)$ and $\mathrm{Sp}(m)$, which are in fact complex manifolds, thus complex geometry can be used to study them. The book [Huy05] provides a general introduction into the subject.
- **Ricci-flat holonomy groups** are $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, G_2 and $\mathrm{Spin}(7)$, which are all important in compactification issues of string and M-theory.
- **Exceptional holonomy groups** are G_2 and $\mathrm{Spin}(7)$, which fall out of the 4-periodic pattern of the possible Riemannian holonomy groups in any dimension. First examples of these manifolds were constructed by Joyce via resolving certain singularities in orbifold constructions, cf. [Joy00, chps. 10-15].

Furthermore, the restricted holonomy group $\mathrm{Hol}^0(g)$ of any Riemannian manifold (M, g) is a product of the groups from Berger's classification and of the holonomy groups of symmetric spaces. Those are known from Cartan's classification of symmetric spaces, which is a directly connected to the classification of simple Lie algebras and a rather lengthy issue, see [Hel01, chp. 10].

5.6. Kähler manifolds and complex index notation

The traditional definition of a complex manifold given in app. A is quite awkward to use, when the complex structure of the manifold is itself the object of interest. Let M be a $2n$-dimensional smooth manifold. An **almost complex structure** is a section $J \in \Gamma\big(\mathrm{End}(\mathrm{T}M)\big)$, which provides in each fiber an endomorphism $J_p \in \mathrm{End}(\mathrm{T}_pM)$, such that $J_p \circ J_p = -\operatorname{Id}_{\mathrm{T}_pM}$ holds. This reproduces the property $\mathrm{i} \cdot \mathrm{i} = -1$ of the imaginary number i for complex numbers. Accordingly, J gives each (real) tangent space T_pM the structure of a complex vector space. Given two vector fields $V, W \in \mathfrak{X}(M) = \Gamma(\mathrm{T}M)$, define the **Nijenhuis tensor**

$$\mathrm{N}_J(V, W) := [V, W] + J\big([JV, W] + [V, JW]\big) - [JV, JW].$$

An almost complex structure is called a **complex structure**, if the associated Nijenhuis tensor vanishes, i.e. $\mathrm{N}_J(V, W) = 0$ for all vector fields. Thus, the Nijenhuis tensor essentially measures the deviation of an almost complex structure J from a complex structure. The pair (M, J) is called a **complex manifold**, and it can be shown that this definition is equivalent to the one given in app. A.

Now, let (M, J) be a complex manifold and g be a Riemannian metric on M. If $g(V, W) = g(JV, JW)$ holds for all vector fields $V, W \in \mathfrak{X}(M)$, the metric g is said to be a **Hermitian metric**. In index notation the complex structure J is represented as J_a^b, and the hermiticity condition of the metric reads

$$g_{ab} = J_a^c J_b^d g_{cd}.$$

This provides a natural compatibility condition between the Riemannian metric g and the complex structure J. Given a Hermitian metric g, one defines the **Hermitian form** ω by $\omega(V, W) := g(JV, W)$ for all vector fields $V, W \in \mathfrak{X}(M)$.[e] The metric g is called a **Kähler metric** if $\mathrm{d}\omega = 0$, and (M, J, g) is a **Kähler manifold** if g is a Kähler metric with respect to the complex structure J. In this context ω is called the **Kähler form**, such that a Kähler manifold is completely specified by (M, J, g, ω). As mentioned before, a Kähler metric is the natural complex analogon to a Riemannian manifold.[f] Using the frame bundle approach, the manifolds tangent bundle $\mathrm{T}M$ can be understood as a principal $\mathrm{U}(n)$-bundle.

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[e] There is no conceptual relation between the Hermitian form ω and the connection form ω, despite using the same letter for both objects.

[f] In this context, one should keep in mind, that $\mathrm{SO}(n)$ is just the selection of one of the two connection components of $\mathrm{O}(n)$, as $\det\big(\mathrm{SO}(n)\big) = \{1\}$ and $\det\big(\mathrm{O}(n)\big) = \{\pm 1\}$. For the complex groups $\det\big(\mathrm{U}(n)\big) = \mathrm{U}(1)$ is a smooth 1-dimensional manifold, but $\det\big(\mathrm{SU}(n)\big) = \{1\}$. Thus, the algebraic relation between $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ is quite different from the relation of $\mathrm{O}(n)$ to $\mathrm{SO}(n)$. This serves as a motivation, why Kähler (instead of Calabi-Yau) manifolds are the natural complex analogon to Riemannian manifolds.

To express the complex nature of a Kähler manifold in the index notation itself, for a given (complex-valued) tensor $T \in \Gamma(\mathrm{T}M^{\otimes p} \otimes \mathrm{T}^*M^{\otimes q})$ with indices $T^{ab\dots}_{cd\dots}$ define for each index

$$\left.\begin{aligned} T^{\alpha\dots}_{\dots} &:= \tfrac{1}{2}\left(T^{a\dots}_{\dots} - \mathrm{i}J^a_j T^{j\dots}_{\dots}\right) \\ T^{\bar{\alpha}\dots}_{\dots} &:= \tfrac{1}{2}\left(T^{a\dots}_{\dots} + \mathrm{i}J^a_j T^{j\dots}_{\dots}\right) \end{aligned}\right\} \qquad T^{a\dots}_{\dots} = T^{\alpha\dots}_{\dots} + T^{\bar{\alpha}\dots}_{\dots}$$

Using this notation, it follows $\delta^b_a = \delta^\beta_\alpha + \delta^{\bar{\beta}}_{\bar{\alpha}}$ and the almost complex structure can be represented as $J^b_a = \mathrm{i}\delta^\beta_\alpha - \mathrm{i}\delta^{\bar{\beta}}_{\bar{\alpha}}$. Essentially, the complex structure acts on the tensor indices $\alpha, \beta, \dots$ by multiplication with i and on $\bar{\alpha}, \bar{\beta}, \dots$ with $-$i.

This J-dependent splitting of indices simplifies notation dramatically: Given a Kähler-manifold (M, J, g) the hermiticity of the Riemannian metric g can be expressed as $g_{ab} = g_{\alpha\bar{\beta}} + g_{\bar{\alpha}\beta}$, which in particular implies $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$. The Kähler form ω takes the form $\omega_{ac} = J^b_a g_{bc}$ in the complex index notation. Furthermore, the Riemannian curvature tensor can be written as

$$R^a{}_{bcd} = R^\alpha{}_{\beta\gamma\bar{\delta}} + R^\alpha{}_{\beta\bar{\gamma}\delta} + R^{\bar{\alpha}}{}_{\bar{\beta}\gamma\delta} + R^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}\delta}.$$

Using the usual symmetries of the Riemannian curvature tensor as stated in sec. 5.4, it can be shown that the Riemannian curvature tensor of a Kähler manifold—called the **Kähler curvature (tensor)** for short—in fact only depends on the components $R^\alpha{}_{\beta\gamma\bar{\delta}}$. The Ricci curvature $R_{ab} = R^p{}_{apb}$ simplifies to

$$R_{ab} = R^\alpha{}_{\beta\alpha\bar{\delta}} + R^{\bar{\alpha}}{}_{\bar{\beta}\bar{\alpha}\delta},$$

which implies $R_{ab} = R_{\alpha\bar{\beta}} + R_{\bar{\alpha}\beta}$ and $R_{\alpha\beta} = R_{\bar{\alpha}\bar{\beta}} = 0$. Analogously to the Kähler form ω, one defines the **Ricci form**

$$\rho_{ac} := J^b_a R_{bc} = \mathrm{i}R_{\alpha\bar{\beta}} - \mathrm{i}R_{\bar{\alpha}\beta},$$

which gives a real differential form on M. The Ricci curvature is recovered from the Ricci form via $R_{ab} = \rho_{ac} J^c_b$.

In particular, the Ricci form is a closed $(1,1)$-form, thus the (singular) cohomology class $[\rho] \in H^2(M;\mathbb{R})$ only depends on the complex structure J of M, which is in fact equal to the first Chern class $2\pi c_1(M)$ by means of Chern-Weil theory.

5.7. Yau's theorem and Calabi-Yau manifolds

The preceding chapter leaves open the question which (1,1)-forms can actually be Ricci forms of a compact Kähler manifold (M, J, g, ω). In 1954 Calabi made the bold conjecture that for any real, closed $(1,1)$-form ρ', satisfying the topological condition $[\rho'] = 2\pi c_1(M)$, there should exist an unique Kähler metric g' on M, such that ρ' is the Ricci form of g' and $[\omega] = [\omega'] \in H^2(M;\mathbb{R})$ holds for the corresponding Kähler forms. Calabi was able to show the uniqueness of such a metric g', but it took until 1976 for a complete proof of the existence. The Chinese mathematician Yau eventually showed that the original statement could be rephrased into a certain kind of second order, non-linear partial differential equation for the metric, called the Monge-Ampère equations. Since partial differential equations in general (and those kind in particular) are difficult to solve, it took over twenty years for a complete proof. Joyce follows the original proof of what is nowadays known as **Yau's theorem** in a very readable manner, see [Joy00, chp. 5] or [Joy07, chp. 6].

Yau's theorem is very important in the context of Riemannian holonomy groups. Let M be a compact Kähler manifold with vanishing first Chern class $c_1(M) = c_1(\mathrm{T}M) = 0$. Then by Yau's theorem the choice $\rho' = 0$ of the closed (1,1)-form provides the existence of a Ricci-flat Kähler metric g' on M. Ricci-flat Kähler metrics in general have the restricted holonomy group $\mathrm{Hol}^0(g) \subseteq \mathrm{SU}(m)$, and if the manifold is irreducible, Berger's classification yields either $\mathrm{Hol}^0(g) = \mathrm{SU}(m)$ in case of a Calabi-Yau manifold or $\mathrm{Hol}^0(g) = \mathrm{Sp}(k)$ for a Hyperkähler manifold. In essence, Yau's theorem provides plenty of examples for Calabi-Yau and Hyperkähler manifolds of $\mathrm{SU}(m)$ or $\mathrm{Sp}(k)$ holonomy.

5.8. Wang's theorem and parallel spinors

The holonomy of a Riemannian manifold has direct consequences for the spinors it admits. Let (M, g) be a Riemannian spin manifold and $S \xrightarrow{\tilde{\pi}} M$ be a (real) spinor bundle associated to the tangent bundle $\mathrm{T}M \xrightarrow{\pi} M$. Note that for any spinor $\psi \in \Gamma(S)$ and vector field $X \in \mathfrak{X}(M) = \Gamma(\mathrm{T}M)$ there is an operation $X \cdot \psi$ induced by (point-wise) Clifford multiplication. Furthermore, there is a natural lifting of the tangent bundle's canonical Levi-Civita connection $\nabla : \Gamma(\mathrm{T}M) \longrightarrow \Gamma(\mathrm{T}^*M \otimes \mathrm{T}M)$ to the spinor bundle, such that one has a natural covariant derivative

$$\nabla : \Gamma(S) \longrightarrow \Gamma(\mathrm{T}^*M \otimes S).$$

This naturally connects Riemannian geometry to the spinor theory. A spinor field $\psi \in \Gamma(S)$ satisfying

$$\nabla_X \psi = \lambda X \cdot \psi$$

for any vector field $X \in \mathfrak{X}(M)$ is called a (conformal) **Killing spinor**. In particular, for $\lambda = 0$ this is called a **parallel spinor**, which satisfies $\nabla_X \psi = 0$ for any $X \in \mathfrak{X}(M)$ and is thus covariantly constant. The latter case will be quite important in the context of Calabi-Yau compactifications, see sec. 8.8.

In extension to Berger's classification of Riemannian holonomy groups, Wang could determine the number of linearly independent parallel spinors. Let (M, g) be a complete, simply-connected, irreducible Riemannian spin manifold of dimension n. If M admits a non-vanishing parallel spinor, then the metric g is Ricci-flat, see [Hit74, thm. 1.2]. However, depending on the dimension n and holonomy, this result can be refined, see [Wan89]. Let N denote the dimension of the space of parallel spinors on M. If (M, g) is non-flat and $N > 0$, then on of the following five cases holds:

(1) $\mathrm{Hol}(g) = \mathrm{SU}(2m)$ and $N = (2, 0)$ for $n = 4m$, i.e. even-dimensional Calabi-Yau manifolds admit up to two linearly independent parallel spinors of same chirality.
(2) $\mathrm{Hol}(g) = \mathrm{SU}(2m+1)$ and $N = (1, 1)$ for $n = 4m+2$, i.e. odd-dimensional Calabi-Yau manifolds admit up to two linearly independent parallel spinors of opposite chirality.
(3) $\mathrm{Hol}(g) = \mathrm{Sp}(k)$ and $N = (k+1, 0)$ for $n = 4k$, which holds for Hyperkähler manifolds.
(4) $\mathrm{Hol}(g) = \mathrm{G}_2$ and $N = 1$ for $n = 7$, admitting a non-chiral parallel spinor.[g]
(5) $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ and $N = (1, 0)$ for $n = 8$, admitting a chiral parallel spinor.

Those numbers N are listed in tab. 5.1 and have a direct effect on the level of supersymmetry, as will be investigated in chap. 8. Since one has to make up for six unobservable dimensions in a 10d theory (i.e. superstrings or heterotic strings) and seven hidden dimensions in a 11d theory (i.e. M-theory), the Calabi-Yau and G_2-holonomy manifolds are of special interest. In particular, the dimension N of the space of parallel spinors has direct implications for the maximal possible level of supersymmetry, allowing only for $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetry in case of the Calabi-Yau compactifications.

5.9. Dirac operators and Dirac bundles

There is another object linking Riemannian and spinor geometry. Suppose $S \xrightarrow{\pi} M$ is a spinor bundle over a Riemannian manifold M furnished with a Riemannian connection ∇. The **Dirac operator** is the canonical first-order differential operator point-wise defined by

$$\begin{aligned} \not{D} : \Gamma(S) &\longrightarrow \Gamma(S) \\ \sigma &\mapsto \sum_{j=1}^{n} e_j \cdot \nabla_{e_j} \sigma \qquad \rightsquigarrow \qquad \underbrace{\not{\partial}\psi = \gamma^\mu \partial_\mu \psi}_{\substack{\text{physical notion for a} \\ \text{fixed flat space-time}}} \end{aligned}$$

[g]This case has gained much interest in the context of smooth M-theory compactification, as it allows for only 10d $\mathcal{N} = 1$ supersymmetry—effectively removing another undesired degree of freedom (choice of the level of supersymmetry).

at $p \in M$, where $e_1, \ldots, e_n$ is an orthonormal basis of $\mathrm{T}_p M$, where "$\cdot$" denotes the usual Clifford multiplication. The Dirac operator is usually regarded as the square root of the Laplacian Δ or D'Alambert operator $\Box$, depending on the space-time signature, which is easily proved from the squaring properties of the generators of the underlying Clifford algebra. In particular, let $S = S^+ \oplus S^-$ be a (s)pinor bundle admitting a chirality splitting, then the Dirac operator interchanges the chiralities, i.e.

$$\not{D} : \Gamma(S^{\pm}) \longrightarrow \Gamma(S^{\mp}).$$

Finally a **Dirac bundle** over a Riemannian manifold is a spinor bundle with two additional properties: The first property is that Clifford multiplication by unit vectors in the tangent bundle TM be orthogonal. The second requirement is that the covariant derivative on S be a module derivation. There is an inner product on the space of spinors $\Gamma(S)$ induced from the point-wise inner product $\langle ., . \rangle$ by setting

$$(\sigma_1, \sigma_2) := \int_M \langle \sigma_1, \sigma_2 \rangle.$$

The Dirac operator of any Dirac bundle over a Riemannian manifold is formally self-adjoint with respect to this inner product, i.e. $(\not{D}\sigma_1, \sigma_2) = (\sigma_1, \not{D}\sigma_2)$. Proofs and further properties of the Dirac operator can be found in [LM89, II.§5] and [Jos95, §3.4]. In particular, [LM89, II.§6] shows how the Euler characteristic is expressed as the index of the Dirac operator, which will be used in chap. 8.

Part II

Supersymmetry, Superstrings and Heterotic Compactification

CHAPTER 6

Supersymmetry and Supergravity

The standard model of particle physics provides an ample understanding for the experimental data generated by modern particle accelerators—that is pre-LHC status. However, it leaves open quite a number of deep conceptual questions, foremost the hierarchy problem: "Why is the Higgs particle so much lighter than any other GUT mass or the Planck mass?" The standard model also does not provide an (microscopic) explanation of the smallness of the cosmological constant, which describes the vacuum energy. Besides that, the standard model is highly unsatisfactory from the technical side, as numerous infinities (UV divergences) arising in actual calculations have to be dealt with by using renormalization techniques. The former two issues can be addressed by introducing the concept of supersymmetry, which is also a crucial ingredient of the superstring theories that aim to get rid of the mentioned divergences. Supersymmetry also leads naturally to gravity when the symmetry is localized, which is called supergravity. All of this entitles a short summary of the basic concepts of supersymmetry (SUSY) and supergravity (SUGRA). Standard references on the physical aspects supersymmetry are the books [Wes90] and [Wei00], whereas the mathematical aspects can be found in [DEF$^+$99, I-Supersymmetry], [Var04] and [Fre99]. A particularly noteworthy source of information regarding supersymmetry and supergravity is the book [GGRS83].

6.1. Minkowski space and Poincaré group

The modern quantum theoretic picture of the world teaches the existence of two fundamentally different types of objects: fermions and bosons, which are distinguished by their statistical behavior. The **fermions** tend to avoid each other because they cannot occupate the same quantum state, and are described by particles of half-integer spin $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. All known fundamental matter constituents are fermions **Bosons**, on the other hand, can accumulate without any restrictions regarding their possible quantum states and are integer spin particles ($s = 0, 1, 2, 3, \ldots$). Some of them mediate the four known fundamental interactions according to the quantum field description of the standard model, which incorporates Einstein's ideas of special relativity.

The stage, on which ordinary quantum field theory usually unfolds, is called the (d-dimensional) **Minkowski space-time** $\mathcal{M}^d$, which is an affine space with a Lorentz signature inner product[a] on the translation vector space V. This definition stresses the physical fact of homogeneity, i.e. the empirical fact that there are no distinguished reference points found in the universe. Thus, for two points $p_1, p_2 \in \mathcal{M}^d$, only the difference (the relative translation vector) $v := p_1 - p_2 \in V$ is relevant. The relativistic distance $v^2 = v \cdot v = \langle v, v\rangle = v_\mu v^\mu$ provided by the inner product between two space-time points is the only physically meaningful quantity. As an alternative, the Minkowski space-time $\mathcal{M}^d$ can be regarded as a commutative Lie group, such that the translation vector space V corresponds to the Lie algebra. This particular point of view will be used to construct the Minkowski superspace-time in sec. 6.3.

The (unit's connection component of the) invariance group of this inner product is generated by space-time translations and infinitesimal Lorentz rotations $\mathfrak{o}(V)$, i.e. rotations preserving the inner product of V. This so-called **Poincaré algebra** can be written as

$$\mathfrak{p}^d := V \oplus \mathfrak{o}(V) \cong \mathbb{R}^{d-1,1} \oplus \mathfrak{o}(d-1, 1).$$

[a]From now on the Lorentz or Minkowski signature will always refer to the "mostly plus" signature $(d-1, 1)$.

Translation generators—called momenta—are usually denoted by P_μ, whereas the **Lorentz rotation generators** of $\mathfrak{o}(d-1,1)$ are expressed via the antisymmetric $M_{\mu\nu} = -M_{\nu\mu}$, such that the Lorentz algebra is explicitly given by the relations

$$\begin{aligned}
&\text{translations:} && [P_\mu, P_\nu] = 0\\
&\text{Lorentz transf.:} && [M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}\\
&\text{mixed :} && [M_{\mu\nu}, P_\rho] = \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu.
\end{aligned}$$

The non-vanishing of $[M_{\mu\nu}, P_\rho]$ indicates a non-trivial intertwinement of the translations and rotations, which is expressed by a semi-direct product at group level. The fine point is again—just as for the Lorentz group $\mathcal{L}$ discussed at the end of chap. 4—to distinguish between the Poincaré algebra $\mathfrak{p}^d$ and the Poincaré group $\mathcal{P}^d$. The quantum behavior (uncertainty principle, compatible observables, etc.) is encoded in the relations of the algebra, whereas the finite transformations are found in the Poincaré group $\mathcal{P}^d$, which is defined to be the adjointment of Lorentz transformations $\mathcal{L}$ (which keep the inner product invariant) and translations V, i.e. the semi-direct product (see sec. A.1)

$$\mathcal{P}^d := V \ltimes \mathcal{L}^d \cong \mathbb{R}^{1,d-1} \ltimes \mathrm{O}(1, d-1), \tag{6.1}$$

such that $\mathfrak{p}^d = \operatorname{Lie}\mathcal{P}^d$ holds. However, since $\mathcal{P}^d$ is not (simply-)connected the converse is not true, it follows $\mathcal{P}^d \neq \exp(\mathfrak{p}^d)$, since a Lie algebra at most describes the unit's connection component. For this reason, the restricted (proper, orthochronous) Poincaré covering group $\tilde{\mathcal{P}}_+^{\uparrow,d} := \exp(\mathfrak{p}^d) = \mathbb{R}^{1,d-1} \ltimes \mathrm{Spin}^0(1, d-1)$ is used, which covers the **restricted Poincaré group**

$$\mathcal{P}_+^{\uparrow,d} := \mathbb{R}^{d-1,1} \ltimes \mathrm{SO}^0(d-1,1).$$

The equivalences at algebra level and coverings, inclusions, etc. for the underlying groups can be depicted as follows:

$$\begin{aligned}
\text{algebra:}\quad & \operatorname{Lie}\mathcal{P}^d = \operatorname{Lie}\tilde{\mathcal{P}}^d = \operatorname{Lie}\tilde{\mathcal{P}}_+^{\uparrow,d} = \operatorname{Lie}\mathcal{P}_+^{\uparrow,d}\\
&\cong \mathbb{R}^{d-1,1} \oplus \mathfrak{o}(d-1,1) \cong \mathbb{R}^{d-1,1} \oplus \mathfrak{pin}(d-1,1)\\
&\cong \mathbb{R}^{d-1,1} \oplus \mathfrak{spin}(d-1,1) \cong \mathbb{R}^{d-1,1} \oplus \mathfrak{so}(d-1,1)
\end{aligned}$$

group:

$$\begin{tikzcd}
\mathcal{P}^d = \mathbb{R}^{d-1,1} \ltimes \mathrm{O}(d-1,1) & \mathbb{R}^{d-1,1} \ltimes \mathrm{Pin}(d-1,1) = \tilde{\mathcal{P}}^d \arrow[l, two heads, "2:1"'] \\
\mathcal{P}_+^{\uparrow,d} = \mathbb{R}^{d-1,1} \ltimes \mathrm{SO}^0(d-1,1) \arrow[u, hook, "\text{subgroup}"] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{R}^{d-1,1} \ltimes \mathrm{Spin}^0(d-1,1) = \tilde{\mathcal{P}}_+^{\uparrow,d} \arrow[l, two heads, "2:1"] \arrow[u, hook, "\text{subgroup}"']
\end{tikzcd}$$

This should be compared to the corresponding diagram for the Lorentz group in sec. 4.8. Both the (restricted) Poincaré group $\mathcal{P}^d$ and its covering group $\tilde{\mathcal{P}}^d$ have real dimension

$$\dim\tilde{\mathcal{P}}^d = \dim\mathcal{P}^d = \dim V + \dim\mathcal{L}^d = d + \frac{1}{2}d(d-1) = \frac{1}{2}d(d+1).$$

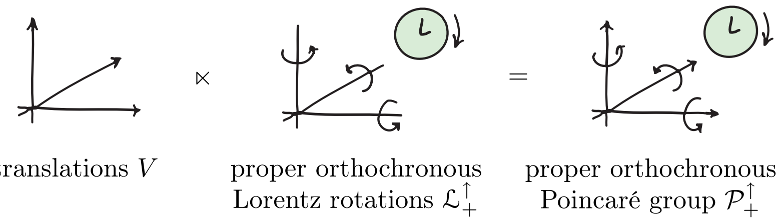

FIGURE 6.1. Illustration of the definition of the restricted (proper, orthochronous) Poincaré group.

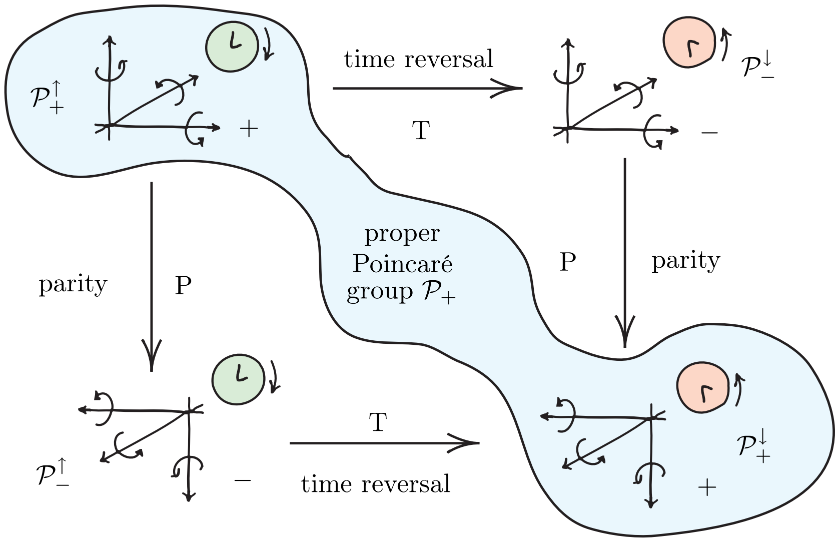

FIGURE 6.2. Connection components of the full Poincaré group.

Note that one can treat the Minkowski space-time $\mathcal{M}^d$ as the coset space $\mathcal{P}^d/\mathcal{L}^d$ or $\mathcal{P}_+^{\uparrow,d}/\mathcal{L}_+^{\uparrow,d}$, i.e. the Minkowski space-time arises naturally for the given symmetry groups.

Elementary quantum particles are identified with the irreducible, unitary, finite-dimensional representations of the Poincaré covering group $\tilde{\mathcal{P}}^d$. Usually $\tilde{\mathcal{P}}_+^{\uparrow,d} \times G$ is considered in actual physical theories instead of $\tilde{\mathcal{P}}^d$, where G is a further internal compact symmetry Lie group. The neglected connection components of $\tilde{\mathcal{P}}^d$ are considered separately by studying the transformation behavior under spatial reflection (P) and time reversal (T). Note that the internal symmetry algebra has no non-trivial relations to the Poincaré algebra, i.e. it is a direct product instead of a semidirect product.

Naturally, the question arises whether this quantum field invariance group could be enlarged in a non-trivial way by considering larger simple Lie groups that contain $\tilde{P}^d \times G$ as a subgroup. In 1967 Coleman and Mandula proved their famous "no-go-theorem" (see [CM67]) that such an extension would only yield trivial physics. The central problem, which is of rather mathematical nature, can be understood as follows: In general, there are no unitary, finite-dimensional representations for a non-compact group. But from the physical point of view, a non-unitary representation would imply either a continua of elementary particle masses or the association of infinite many particle states to a single irreducible representation. It is a rather fortunate fact, that the non-compact Poincaré covering group $\tilde{\mathcal{P}}^d$ indeed possesses unitary, finite-dimensional, irreducible representations, which are explicitly described in a physical context in [SU01].

6.2. Superalgebras, Superspace and Superfields

Supersymmetry was originally proposed in 1973 by Wess and Zumino, cf. [WZ74]. In fact, the supersymmetry algebra had been written down in the late 1960s by Soviet theorists Gol'fand and Likhtman (albeit not in the context of particle physics), but due to the political situation knowledge of this discovery did not come to a broader audience. The notion of Lie algebra was successfully expanded to include fermionic generators, i.e. generators satisfying anti-commutation relations and therefore making it possible to construct a symmetry between particles of different statistics. The result of Coleman and Mandula is then circumvented due to the Haag-Łopuszański-Sohnius theorem (see [HŁS75]).

Due to the anti-commuting nature any square (or higher power) of a fermionic quantity vanishes. Thus, for the fermionic generators the infinitesimal and finite transformations are simply related in a suitable representation, see below. The important effect of applying a fermionic generator on a given elementary particle state is a change by spin $\frac{1}{2}$, which by the spin-statistics theorem is equivalent to turning a boson into a fermion and vice versa. Those ideas can be formalized as follows:

- The **component approach** is technically simple as it builds directly onto the established quantum field formalism, i.e. one puts a number of bosonic and fermionic fields in a suitable Lagrangian. However, one has to precisely fit the transformation behavior of each field in order to provide invariance under supersymmetric transformations. Since the component fields have a direct particle interpretation, the physical properties of a supersymmetric theory are very accessible in terms of component fields.
- In the **superspace formalism** the underlying space-time is turned into a superspace-time, i.e. anti-commuting coordinates are added to the ordinary space-time coordinates. Fields defined on such a superspace-time are manifestly invariant under SUSY transformations, which is the central benefit of this approach. Furthermore, the superspace formalism enjoys a rather rigorous mathematical formulation, see [Var04], [Fre99] and [DEF$^+$99]. By expanding in powers of the anti-commuting coordinates—which terminates after linear order—contact is made with the corresponding component field formalism.

In the following sections, the superspace approach is introduced in order to provide a mathematically rigorous basis. However, since a formulation of supergravity in terms of superfields is complicated—albeit possible—the component fields are used in the later sections.

A **Lie superalgebra** $\mathfrak{s}$ is a $\mathbb{Z}_2$-graded vector space $\mathfrak{s} = \mathfrak{s}^0 \oplus \mathfrak{s}^1$, where $\mathfrak{s}^0$ is the even part and $\mathfrak{s}^1$ the odd part, equipped with a super-skew-symmetric mapping $[.,.]_\mathfrak{s} : \mathfrak{s} \otimes \mathfrak{s} \longrightarrow \mathfrak{s}$, i.e. $[X,Y]_\mathfrak{s} = (-1)^{xy}[Y,X]_\mathfrak{s}$, that satisfies the **Jacobi superidentity**

$$(-1)^{zy}\big[X,[Y,Z]_\mathfrak{s}\big]_\mathfrak{s} + (-1)^{xy}\big[Y,[Z,X]_\mathfrak{s}\big]_\mathfrak{s} + (-1)^{yz}\big[Z,[X,Y]_\mathfrak{s}\big]_\mathfrak{s} = 0,$$

where the lower-case letters denote the grading of the respective elements, i.e. $\deg X = 0$ for $X \in \mathfrak{s}^0 \subset \mathfrak{s}$ and $\deg X = 1$ for $X \in \mathfrak{s}^1$. The mapping $[.,.]_\mathfrak{s}$ will be referred to as the **Lie superbracket**. If all signs in the definition are neglected, one obtains an ordinary Lie algebra. In the physical context, the even part is identified with bosonic components whereas the odd part refers to fermionic behavior, so in the physical literature one might find $\mathfrak{s} = \mathfrak{s}_\mathrm{B} \oplus \mathfrak{s}_\mathrm{F}$. In particular, the general behavior as a $\mathbb{Z}_2$-graded algebra implies

$$[X,Y]_\mathfrak{s} \in \mathfrak{s}_{x+y \bmod 2} \subset \mathfrak{s} \qquad \rightsquigarrow \qquad \begin{aligned} [\mathrm{B},\mathrm{B}]_\mathfrak{s} &= [\mathrm{B},\mathrm{B}] = \mathrm{B} \\ [\mathrm{F},\mathrm{F}]_\mathfrak{s} &= \{\mathrm{F},\mathrm{F}\} = \mathrm{B} \\ [\mathrm{B},\mathrm{F}]_\mathfrak{s} &= [\mathrm{B},\mathrm{F}] = \mathrm{F} \end{aligned}$$

when considering Lie superbrackets of bosonic (B, even) and fermionic (F, odd) type. The relevant extension of the Poincaré algebra $\mathfrak{p}^d$ will be constructed in the next section.

In order to generalize the notion of a manifold, one has to adjoin anti-commuting coordinates. The **prototype superspace** $\mathbb{R}^{p|q}$ has p even real coordinates $(x_1,\dots,x_p)$ and q odd (anti-commuting) Grassmann coordinates $(\theta_1,\dots,\theta_p)$, such that $\dim_\mathbb{R} \mathbb{R}^{p|q} = p+q$. The even / odd separation is only relevant in the context of functions defined on $\mathbb{R}^{p|q}$, called superfields, but can also be understood from the fact, that $\mathbb{R}^{p|q}$ forms a Lie superalgebra with all possible superbrackets vanishing.

A **superfield** on the superspace $\mathbb{R}^{p|q}$ is an element $\Phi \in C^\infty(\mathbb{R}^p) \otimes \Lambda^\bullet(\theta_1, \ldots, \theta_q)$, i.e. a collection of $\dim_\mathbb{R} \Lambda^\bullet(\theta_1, \ldots, \theta_q) = 2^q$ component functions $\phi_i \in C^\infty(\mathbb{R}^p)$, such that

$$\begin{aligned}\Phi \underbrace{(x_1, \ldots, x_p, \theta_1, \ldots, \theta_q)}_{\substack{p+q \text{ superspace coordinates;} \\ p \text{ even, } q \text{ odd}}} &= \phi(x_1, \ldots, x_p) \\ &+ \theta_1 \phi_1(x_1, \ldots, x_p) + \ldots + \theta_q \phi_q(x_1, \ldots, x_p) \\ &+ \theta_1\theta_2\phi_{12}(x_1, \ldots, x_p) + \ldots + \theta_{q-1}\theta_q\phi_{q-1,q}(x_1, \ldots, x_p) \\ &+ \theta_1\theta_2\theta_3\phi_{123}(x_1, \ldots, x_p) + \ldots \\ &+ \ldots \\ &+ \theta_1\theta_2\cdots\theta_q\phi_{123\ldots q}(x_1, \ldots, x_p).\end{aligned}$$

Since any differentiable manifold locally looks like $\mathbb{R}^p$, the above construction can be generalized to **supermanifolds** $M^{p|q}$, which locally look like $\mathbb{R}^{p|q}$. However, this definition is not precise,[b] but this shall not be investigated further. One can introduce **(p, q)-superalgebra bundles** in the same fashion, i.e. the fibers are superalgebras $\mathbb{R}^{p|q}$. There is also a mapping

$$\Pi : \mathbb{R}^{p|q} \longrightarrow \mathbb{R}^{q|p}$$

which exchanges the even and odd coordinates of the prototype superspace, which generalizes to superalgebra bundles. A simple example of a $(0, n)$-superbundle is the **odd tangent bundle** $\Pi(\mathrm{T}M) \xrightarrow{\pi} M$ for M being a smooth n-dimensional manifold, which is also an example of a non-trivial supermanifold if one forgets about the bundle structure.

6.3. Minkowski superspace and Poincaré supergroup

For the general construction of the d-dimensional Minkowski superspace-time, the minimal real spin representation S of $\mathrm{Spin}(d-1, 1)$ is required—however, several copies of it are also admissible for S. Those representations are listed in tab. 6.1 for all relevant dimensions. The number of copies of the minimal real spin representation in S is usually denoted by $\mathcal{N}$. Due to the Lorentz signature of the considered situation, there exist symmetric, equivariant pairings for both the spin representation S and the dual representation S^*,

$$\Gamma : S^* \otimes S^* \longrightarrow V \qquad \text{and} \qquad \tilde{\Gamma} : S \otimes S \longrightarrow V,$$

yielding the underlying vector representation V in every dimension d—which is quite a unique feature of the Lorentz signature. Let $\{P_\mu\}$ be a basis for the vector representation V, $\{Q^a\}$ be a basis for the spinor representation space S and $\{Q_a\}$ for the dual representation S^*. Then both pairings can be expressed as

$$\Gamma(Q_a, Q_b) = \Gamma^\mu_{ab} P_\mu \qquad \text{and} \qquad \tilde{\Gamma}(Q^a, Q^b) = \tilde{\Gamma}^{\mu ab} P_\mu.$$

In dimensions $d \not\equiv 2, 6 \bmod 8$ there also exists a duality pairing $\epsilon : S \otimes S \longrightarrow \mathbb{R}$, which is skew-symmetric for $d \equiv 3, 4 \bmod 8$ and symmetric for $d \equiv 1, 5, 7, 8 \bmod 8$. Thus, it induces an isomorphism $S^* \cong S$, such that in terms of components $\epsilon(Q^a, Q^b) = \epsilon^{ab}$ the two Γ-pairings are related via

$$\tilde{\Gamma}^{\mu ab} = \Gamma^\mu_{a'b'} \epsilon^{aa'} \epsilon^{bb'}. \tag{6.2}$$

For the remaining dimensions $d \equiv 2, 6 \bmod 8$ similar pairings relate the $S^\pm$-splitting, which replaces S and S^* in those cases, see tab. 6.1. It can then be shown, that the coefficients satisfy the general Clifford algebra relations

$$\Gamma^\mu_{ab} \tilde{\Gamma}^{\nu bc} + \Gamma^\nu_{ab} \tilde{\Gamma}^{\mu bc} = 2\eta^{\mu\nu} \delta^c_a.$$

[b]One could ask, if there are obstructions to adjoining q anti-commuting coordinates or other ambiguities. The usual mathematical definition is as follows: A **supermanifold** $M^{p|q}$ is a topological space $(M, \mathfrak{T})$ together with a sheaf of superalgebras $C^\infty(M^{p|q})$ locally isomorphic to $C^\infty(\mathbb{R}^p) \otimes \Lambda^\bullet(\theta_1, \ldots, \theta_q)$. This definition automatically provides what mappings between supermanifolds are, etc. As mentioned before, sheaves are a mathematical tool to generalize local constructions in the context of mappings to global ones. For a thorough introduction to supermanifolds see [DEF+99, I-Supersymmetry, chp. 2].

d	minimal rep. type	duality	complexification	dim S	$\mathcal{N}_{\max}$
1	S real	$S^* \cong S$		1	32
2	S^+, S^- real	$(S^+)^* \cong S^-$		1	32
3	S real	$S^* \cong S$		2	16
4	$S' \sim S''$ complex	$S' \cong \bar{S}''$	$S_{\mathbb{C}} \cong S' \oplus S''$	4	8
5	S quat.	$S^* \cong S$	$S_{\mathbb{C}} \cong S_0 \otimes_{\mathbb{C}} W$	8	4
6	S^+, S^- quat.	$(S^+)^* \cong S^-$	$S^{\pm}_{\mathbb{C}} \cong S^{\pm}_0 \otimes_{\mathbb{C}} W$	8	4
7	S quat.	$S^* \cong S$	$S_{\mathbb{C}} \cong S_0 \otimes_{\mathbb{C}} W$	16	2
8	$S' \sim S''$ complex	$S' \cong \bar{S}''$	$S_{\mathbb{C}} \cong S' \oplus S''$	16	2
9	S real	$S^* \cong S$		16	2
10	S^+, S^- real	$(S^+)^* \cong S^-$		16	2
11	S real	$S^* \cong S$		32	1

TABLE 6.1. The table summarizes the properties of the minimal representations of $\mathrm{Spin}(d-1,1) \cong \mathrm{Spin}(1,d-1)$. In the second column, the representation type (real, complex, quaternionic) is indicated. Of course, a complex representation has an underlying real one under the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$. In all even dimensions a chirality splitting is found for complex spinors (recall the classification of $\mathbb{C}\ell_{2n}$), but not necessarily for the real spinors. Thus, the underlying real representations of the complex (inequivalent) representations S', S'' in $d = 4, 8$ are in fact equivalent, as indicated by $S' \sim S''$. This complex chirality splitting is recovered upon complexification, as indicated in the fourth column. For quaternionic representations in even dimensions (case $d = 6$) the underlying real representations are inequivalent, and the fourth column indicates their structure upon complexification, where W is a 2-dimensional complex vector space with $\mathbb{H}$-action and S_0 is a complex representation of half the quaternionic dimension of S.

Finally, fix a positive cone of time-like vectors $C \subset V$ and require the pairing to satisfy the **positivity condition** $\Gamma(s^*, s^*) \in \overline{C}$ for all $s^* \in S^*$ and $\Gamma(s^*, s^*) = 0 \in \overline{C}$ if and only if $s^* = 0$. This amounts to a choice of a positive time arrow in purely abstract terms, which—using the forthcoming relation (6.4)—has an important physical consequence:

- The energy (expressed by the Hamiltonian) in any supersymmetric theory is always positive. In particular this implies the absence of tachyonic states, i.e. states with negative mass-square.

Using this collection of data, define the Lie superalgebra

$$\mathfrak{m} := \big[V\big]_{\mathrm{B}} \oplus \big[S^*\big]_{\mathrm{F}} \tag{6.3}$$

with even and odd part as indicated, such that the only non-trivial superbracket is given by

$$[Q_a, Q_b]_{\mathfrak{m}} = \{Q_a, Q_b\} = -2\Gamma(Q_a, Q_b) = -2\Gamma^{\mu}_{ab} P_{\mu} \tag{6.4}$$

for any two odd generators $Q_a, Q_b \in S^* = \mathfrak{m}_{\mathrm{F}}$. Note that $V = \mathfrak{m}_{\mathrm{B}}$ is central, i.e. the elements of the vector representation commute with all generators. Naturally, V is interpreted as the translation vector space in the physical context. Let $\exp(\mathfrak{m})$ denote the corresponding universal Lie supergroup and s the (real) dimension of the spin representation S. The underlying supermanifold $\mathcal{M}^{d|s}$ of $\exp(\mathfrak{m})$ can then be written as

$$\mathcal{M}^{d|s} \cong \mathcal{M}^d \times \Pi S^*,$$

where $\mathcal{M}^d$ is the affine Minkowski space-time underlying the translation vector (and representation) space V. Note that the spin representation space S^* has to be changed to anti-commuting structure by Π, since as a representation it is of even character. This supermanifold $\mathcal{M}^{d|s}$ is called the **Minkowski superspace-time**, which adjoins to the ordinary Minkowski space-time a (multiple of the) minimal (real) s-dimensional spin representation as odd coordinates. This provides the stage for supersymmetric theories.

As before, let (x^μ, θ^a) denote the even / odd coordinates on the supermanifold $\mathcal{M}^{d|s}$, which induce coordinate vector fields ∂_μ and $\partial_a := \frac{\partial}{\partial\theta^a}$. Since those vector fields are defined on a Lie supergroup, the notion of left- and right—as invariance defined for vector fields on ordinary Lie groups (see sec. A.17)—can be extended to the vector fields on a Lie supergroup. One introduces the natural left- and right-invariant vector fields

$$\begin{aligned} D_a &:= \partial_a - \Gamma^\mu_{ab}\theta^b\partial_\mu && \text{(left-invariant)} \\ \tau_a &:= \partial_a + \Gamma^\mu_{ab}\theta^b\partial_\mu, && \text{(right-invariant)} \end{aligned}$$

in $\mathfrak{X}(\mathcal{M}^{d|s})$, which after a quick calculation are shown to satisfy the non-trivial brackets

$$\begin{aligned} [D_a, D_b] &= -2\Gamma^\mu_{ab}\partial_\mu \\ [\tau_a, \tau_b] &= +2\Gamma^\mu_{ab}\partial_\mu \end{aligned} \qquad \text{and} \qquad [D_a, \tau_b] = 0,$$

i.e. those left-invariant **supercovariant vector-fields** represent the $\{Q_a, Q_b\}$-bracket under the identification $P_\mu \to \partial_\mu$, whereas the right-invariant ones have the wrong sign. The left-invariant vector fields will be used later on to define certain superfields, see sec. 6.5.

After constructing the stage for manifestly supersymmetric theories—the Minkowski superspace-time $\mathcal{M}^{d|s}$—the corresponding symmetry group has to be defined as well. This is conceptually done in the same fashion as for the ordinary Minkowski space-time, cf. sec. 6.1. The invariance group of the Minkowski superspace-time $\mathcal{M}^{d|s}$ is generated by the **Poincaré superalgebra**

$$\mathfrak{p}^{d|s} := \big[V \oplus \mathfrak{o}(V)\big]_{\mathrm{B}} \oplus \big[S^*\big]_{\mathrm{F}},$$

which is a Lie superalgebra that contains both the ordinary Poincaré algebra $\mathfrak{p}^d$ and the Lie superalgebra $\mathfrak{m}$ as subalgebras. In explicit terms, the superbrackets reduce to[c]

$$\begin{aligned} &\text{translations:} && [P_\mu, P_\nu] = 0 \\ &\text{Lorentz transf.:} && [M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} \\ &\text{SUSY transf.:} && \{Q_a, Q_b\} = -2\Gamma^\mu_{ab}P_\mu \\ &\text{mixed :} && [M_{\mu\nu}, P_\rho] = \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu \\ &&& [P_\mu, Q_a] = 0 \\ &&& [M_{\mu\nu}, Q_a] = 0. \end{aligned}$$

Note in particular, that two successive SUSY transformations—or rather the anti-commutator $Q_aQ_b + Q_bQ_a$—yield a translation, which is later quite important for supergravity.

The **Poincaré supergroup** $\mathcal{P}^{d|s}$ equals the semi-direct product $\exp(\mathfrak{m}) \ltimes \mathrm{O}(V)$, which contains the ordinary Poincaré group $\mathcal{P}^d$ as a subgroup. As before, a restricted Poincaré supergroup $\mathcal{P}_+^{\uparrow,d|s}$ is introduced by using $\mathrm{SO}^0(1, d-1) \subset \mathrm{O}(1, d-1) \cong \mathrm{O}(V)$. Furthermore, there are the covering groups $\tilde{\mathcal{P}}^{d|s} = \exp(\mathfrak{m}) \ltimes \mathrm{Pin}(V)$ for the full Poincaré supergroup and $\tilde{\mathcal{P}}_+^{\uparrow,d|s} = \exp(\mathfrak{m}) \ltimes \mathrm{Spin}^0(V)$ for the restricted one. Similar to the Minkowski space-time $\mathcal{M}^d$, the Minkowski superspace-time has a natural description as the coset space

$$\mathcal{M}^{d|s} \cong \mathcal{P}^{d|s}/\mathcal{L}^{d|s} \cong \tilde{\mathcal{P}}_+^{\uparrow,d|s}/\tilde{\mathcal{L}}_+^{\uparrow,d|s}. \tag{6.5}$$

This shows, that the construction outlined indeed generalizes the concept of the Minkowski space-time in a natural way. Later on this point of view will be used for an explicit description of the SUSY generators and their effects.

[c]Note that in most of the physical literature a factor $-\mathrm{i}$ is used in the definition of the Lie algebra generators, which is canceled by an additional factor i in the exponential function.

6.4. Representations of the Poincaré supergroup

A supersymmetric quantum field theory builds upon the irreducible, unitary, finite-dimensional representations of the covering Poincaré supergroup $\tilde{\mathcal{P}}^{d|s}$ instead of $\tilde{\mathcal{P}}^{d}$. Such a representation breaks up into a finite sum of (physical) representations when restricted to the Poincaré subgroup $\tilde{\mathcal{P}}^{d} \subset \tilde{\mathcal{P}}^{d|s}$. The collection of such particles is called a **supersymmetry multiplet**.

In order to enumerate the physical (irreducible, unitary, finite-dimensional) representations of $\tilde{\mathcal{P}}_{+}^{\uparrow,d|s}$ one has to factorize out translations from the restricted Poincaré group, which yields

$$\tilde{\mathcal{P}}_{+}^{\uparrow,d|s}/V \cong \Pi S^* \ltimes \mathrm{Spin}^0(1, d-1).$$

To enumerate the physical degrees of freedom for a moving particle one has to consider the little group (the stabilizer subgroup of $\Pi S^* \ltimes \mathrm{Spin}^0(1, d-1)$ that keeps the movement invariant), i.e. one has to distinguish between massless and massive particles. This gives the following subgroups:

$$\begin{array}{ll} \text{massive case } (m > 0): & \Pi S^* \ltimes \mathrm{Spin}(d-1) \\ \text{massless case } (m = 0): & \Pi S^* \ltimes \mathrm{Spin}(d-2) \end{array}$$

The general description of the unitary representations of those little groups (see [Wei95, §2.5] for the relation of the little group to the physical states) are rather complicated, see [CCTV06] for a detailed account on the representation theory of such supergroups. The general construction can be outlined as follows: For a given vector (or rather its dual linear functional) $p \in V^*$, a quadratic form q_p on the (dual) spinor representation space S^* is induced via

$$\begin{aligned} \rho : \mathrm{Sym}^2 S^* &\longrightarrow \mathbb{R} \\ s_1^* \otimes s_2^* &\mapsto p\big(\{s_1^*, s_2^*\}\big) \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} q_p : S^* &\longrightarrow \mathbb{R} \\ s_1^* &\mapsto \rho(s_1^*, s_1^*). \end{aligned}$$

Due to the positivity condition, this quadratic form $q_p : S^* \longrightarrow \mathbb{R}$ is negative semi-definite. More precisely, in the massive case it is negative definite (i.e. $\ker q_p = \{0\}$), whereas for the massless case the kernel is non-trivial, such that in general the quotient space $\hat{S} := S^*/\ker(q_p)$ is considered instead.

The required representations are then $\mathbb{Z}_2$-graded $\mathrm{C}\ell(\hat{S}, q_p)$-modules $W = W^0 \oplus W^1$, called supermodules in the mathematical sense and supermultiplets in the physical context, with an intertwining action of the group $\mathrm{Spin}(\hat{S}, q_p)$. Since even and odd parts of a Clifford module are of equal dimension, the construction yields an important physical consequence:

- Every irreducible SUSY representation has as many bosonic degrees of freedom as fermionic ones.

By further considerations, one can also prove the following statement:

- All particles within a supermultiplett have the same mass. This property is rather undesired (since it is not observed in experiment) and forces the use of some kind of SUSY-breaking mechanism to arrive at a realistic spectrum.

The action of the Poincaré supergroup $\tilde{\mathcal{P}}_{+}^{\uparrow,d|s}$ on the superspace $\mathcal{M}^{d|s}$ is given by inverse left multiplication. This will be detailed in explicit terms for $d = 4$ in the following section. In particular, the transformations induced by the fermionic (odd) generators Q_a are called **supersymmetry transformations**. Let $s_{\min} := \dim S_{\min}$ be the dimension of the minimal real spinor representation. A theory invariant under $\tilde{\mathcal{P}}^{d|s}$ for $s = \mathcal{N} s_{\min}$ is called **$\mathcal{N}$-supersymmetric**. If $\mathcal{N} > 1$, this is called **extended supersymmetry**, whereas for $\mathcal{N} = 1$ one calls it **simple** or **minimal supersymmetry**. The case $\mathcal{N} = 0$ corresponds to ordinary non-supersymmetric theories.

From the physical point of view, $\mathcal{N}$ has an upper bound depending on the (super-)spacetime dimension d. This is due to the fact that physical theories are constrained to contain only particles of spin ≤ 2, which was formalized in the Weinberg-Witten theorem, see [WW80]. Thus, one can have up to 32 odd generators in the Poincaré superalgebra, which—depending

4d $\mathcal{N} = 1$ chiral multiplet:

bosons (2b):	2	scalars	scalar superpartners (2b)
fermions (2f):	1	fermion	Weyl spin-$\frac{1}{2}$ fermion (2f)

4d $\mathcal{N} = 1$ SYM multiplet:

bosons (2b):	1	vector	Yang-Mills gauge field (2b)
fermions (2f):	1	gaugino	Majorana spin-$\frac{1}{2}$ fermion (2f)

4d $\mathcal{N} = 1$ SUGRA multiplet:

bosons (2b):	1	graviton	symmetric tensor with fixed trace (2b)
fermions (2f):	1	gravitino	Majorana spin-$\frac{3}{2}$ spinor-vector fermion (2f)

Table 6.2. Massless supermultiplets of the 4-dimensional $\mathcal{N} = 1$ supersymmetry.

4d $\mathcal{N} = 4$ SYM multiplet:

bosons (8b):	1	vector	Yang-Mills gauge field (2b)
	6	scalars	scalar fields (6b)
fermions (8f):	4	gaugini	Majorana spin-$\frac{1}{2}$ fermions (8f)

4d $\mathcal{N} = 4$ SUGRA multiplet:

bosons (16b):	1	graviton	symmetric tensor with fixed trace (2b)
	6	graviphotons	U(1)-Yang Mills gauge fields (12b)
	1	dilaton	scalar (1b)
	1	scalar	(pseudo-)scalar (1b)
fermions (16f):	4	gravitini	Majorana spin-$\frac{3}{2}$ spinor-vector fermions (8f)
	4	dilatini	Majorana spin-$\frac{1}{2}$ fermions (8f)

Table 6.3. Massless supermultiplets of the 4-dimensional $\mathcal{N} = 4$ supersymmetry.

on the (super-)space-time dimension—allows for $\mathcal{N} = 8$ supersymmetry in 4d, but makes $\mathcal{N} = 1$ SUSY the only possibility in 11d. The minimal real spin representations and the possible degrees of supersymmetry are listed in tab. 6.1.

6.5. Supersymmetry in four dimensions

The rather abstract constructions of the previous chapters are now specialized to 4d and $\mathcal{N} = 1$ SUSY, which allows to explain several aspects is more explicit terms. As found in tab. 6.1, in 4 dimensions there are two inequivalent complex representations of $\mathrm{Spin}^0(3,1)$ which have the same underlying real 4d spinor representation S, such that $S \otimes \mathbb{C} \cong S' \oplus S''$ is the ordinary Dirac spinor splitting into two Weyl spinors of opposite chirality. Since the two complex representations S' and S'' are in fact complex conjugates of each other, i.e. $S' \cong \bar{S}''$, it is natural to utilize this structure.

In order to construct the odd part ΠS^* of the Minkowski superspace-time, fix complex odd coordinates θ^1, θ^2 on $\Pi S'^*$ and complex conjugate coordinates $\bar{\theta}^{\dot{1}}, \bar{\theta}^{\dot{2}}$ on $\Pi S''^*$. Upon complexification, the translation vector space V also splits as $V_{\mathbb{C}} \cong S'^* \oplus S''^*$, which can be understood due to the Γ-pairings. The corresponding complex coordinates are induced via

$$x^{a\dot{a}} := \frac{1}{2}(\sigma_\mu)^{a\dot{a}} x^\mu \qquad \text{and} \qquad x^\mu = (\sigma^\mu)_{a\dot{a}} x^{a\dot{a}},$$

10d $\mathcal{N}=1$ chiral multiplet:

bosons (8b):	8	scalars	scalar superpartners (8b)
fermions (8f):	1	fermion	Majorana-Weyl spin-$\frac{1}{2}$ fermions (8f)

10d $\mathcal{N}=1$ SYM multiplet:

bosons (8b):	1	vector	Yang-Mills gauge field (8b)
fermions (8f):	1	gaugino	Majorana-Weyl spin-$\frac{1}{2}$ fermions (8f)

10d $\mathcal{N}=1$ SUGRA multiplet:

bosons (64b):	1	graviton	symmetric tensor with fixed trace (35b)
	1	2-form	antisymmetric tensor (28b)
	1	dilaton	scalar (1b)
fermions (64f):	1	gravitino	L-Majorana-Weyl spin-$\frac{3}{2}$ spinor-vector fermions (56f)
	1	dilatino	R-Majorana-Weyl spin-$\frac{1}{2}$ fermions (8f)

TABLE 6.4. Massless supermultiplets of the 10-dimensional $\mathcal{N}=1$ supersymmetry.

where $\sigma_\mu = (\mathbb{1}, \sigma_i)$ refers to the Pauli matrices. Those complex coordinates are collected as $z := (x, \theta, \bar{\theta})$.

Using the group description (6.5) of the Minkowski superspace-time $\mathcal{M}^{d|s}$ (expressed through the Lie superalgebra (6.3) which generates the "Minkowski supergroup"), one can parameterize it by

$$p(z) = p(x, \theta, \bar{\theta}) = \exp\left(x^{a\dot{b}} P_{a\dot{b}} + \theta^a Q_a + \bar{\theta}^{\dot{a}} \bar{Q}_{\dot{a}}\right).$$

The action of the Poincaré supergroup $\tilde{\mathcal{P}}_+^{\uparrow,d|s}$ on the superspace $\mathcal{M}^{d|s}$ is then given by

$$h(x', \theta', \bar{\theta}') = g^{-1} h(x, \theta, \bar{\theta}) \bmod \mathcal{L}_+^{\uparrow,d},$$

where $g \in \tilde{\mathcal{P}}_+^{\uparrow,d|s}$ is a group element and the identification by $\mathcal{L}_+^{\uparrow,d}$ means to remove all remaining Lorentz rotations. In particular, the fermionic generators yield group elements

$$g(\epsilon, \bar{\epsilon}) := \exp\left(\epsilon^a Q_a + \bar{\epsilon}^{\dot{a}} \bar{Q}_{\dot{a}}\right) \in \tilde{\mathcal{P}}_+^{\uparrow,d|s},$$

where ϵ^a and $\bar{\epsilon}^{\dot{a}}$ are global constant spinors, that parameterize the supersymmetry transformations. The effect on the coordinates of the Minkowski superspace-time $\mathcal{M}^{d|s}$ is

$$\text{SUSY transformation:} \qquad \begin{cases} x'^{a\dot{a}} = x^{a\dot{a}} + \frac{1}{2}\left(\epsilon^a \bar{\theta}^{\dot{a}} + \bar{\epsilon}^{\dot{a}} \theta^a\right) \\ \theta'^a = \theta^a + \epsilon^a \\ \bar{\theta}'^{\dot{a}} = \bar{\theta}^{\dot{a}} + \bar{\epsilon}^{\dot{a}}, \end{cases}$$

where the even and odd coordinates get mixed up. This is not the case for translations or Lorentz rotations, as can be worked out using the Baker-Campbell-Hausdorff formula. Obviously, the SUSY generators yield coordinate transformations on the Minkowski superspace-time $\mathcal{M}^{d|s}$. Further details on the other generators, their explicit representation in terms of partial derivatives, etc. are found in [GGRS83, chp. 3].

Note that the SUSY-parameterizing spinors ϵ^a and $\bar{\epsilon}^{\dot{a}}$ are constant, i.e. the supersymmetry transformation is applied globally in the same manner. In this context, one speaks of **rigid** or **global supersymmetry**, in contrast to the local supersymmetry, which will be introduced in the next section.

As mentioned, superfields allow for an easy formulation of manifestly SUSY-invariant theories. Given a real scalar superfield $\mathcal{F} : \mathcal{M}^{4|4} \longrightarrow \mathbb{R}$ in the 4d $\mathcal{N}=1$ situation, the corresponding component field expansion is

$$\begin{aligned}\mathcal{F}(x,\theta,\bar\theta) = C + \theta^a\chi_a + \bar\theta^{\dot a}\bar\chi_{\dot a} - \theta^2 M - \bar\theta^2\bar M + \underbrace{\theta^a\bar\theta^{\dot a}A_{a\dot a}}_{(\theta\sigma^\mu\bar\theta)A_\mu} \\ - \bar\theta^2\theta^a\lambda_a - \theta^2\bar\theta^{\dot a}\bar\lambda_{\dot a} + \theta^2\bar\theta^2 F,\end{aligned} \quad\rightsquigarrow\quad \begin{array}{ll} 4 & \text{scalars } (C, M, \bar M, F) \\ 4 & \text{Weyl fermions } (\chi, \bar\chi, \lambda, \bar\lambda) \\ 1 & \text{vector } (A)\end{array}$$

i.e. the $\dim_{\mathbb{R}} \Lambda^\bullet(\theta^1, \theta^2, \bar\theta^{\dot 1}, \bar\theta^{\dot 2}) = 2^4 = 16$ real components can be rearranged into 4 scalars, 4 spin-$\frac12$ fermions and 1 vector, yielding eight bosonic and—equally—eight fermionic degrees of freedom. The "square" θ^2 is of course understood as $\theta^2 = \theta^a\theta_a = \epsilon_{ab}\theta^a\theta^b = 2\theta^1\theta^2$, where ϵ_{ab} are the components of the ϵ-pairing mentioned in (6.2), which reduces to the antisymmetric tensor $\epsilon_{ab} = -\epsilon_{ba} = 1$ in this case.

One can consider constrained superfields—foremost the **chiral superfield** Φ, which is defined by the condition $\bar D_{\dot a}\Phi = 0$. This effectively kills the $\bar\theta$-dependence of a general complex superfield, and leaves the much simpler expansion

$$\Phi(x,\theta,\bar\theta) = \Phi(x,\theta) = \phi + \theta^a\psi_a + \theta^2 F \quad\rightsquigarrow\quad \begin{array}{ll} 2 & \text{scalars } (\phi, F) \\ 1 & \text{Weyl fermions } (\psi)\end{array}$$

containing 2 scalar bosons and 1 chiral (left-handed) spin-$\frac12$ Weyl fermion. Actually, the second scalar F is only an auxiliary field, which is present in order to match the number of fermionic degrees of freedom—in the corresponding "superscalar Lagrangian" $\Phi\bar\Phi$ its field equation is $F = 0$. If instead of the $\bar\theta$-dependence the coordinate θ is kept fixed, then one obtains a superfield $\bar\Phi$ of opposite chirality.

The chiral supermultiplet[d] is of utmost importance, since it contains a single (left-handed) spin-$\frac12$ particle as its highest spin state. This allows for the construction of chiral supersymmetric field theories, as required for any approach ultimately leading to the standard model. It can be shown, that in any irreducible supermultiplet there is a single maximum spin component of spin $s + \frac12\mathcal{N}$. Thus, if one seeks for a supermultiplet with a single left-handed spin-$\frac12$ fermion, $\mathcal{N} = 1$ is the highest possible level of supersymmetry. In other words, extended supersymmetry ($\mathcal{N} > 1$) does not allow for the formulation of theories containing chiral spin-$\frac12$ fermions, i.e. no chiral matter.

6.6. The vielbein formalism

Before supergravity can be introduced, one has to understand how spinors can be coupled to the space-time metric. From this point on, the component field description of supersymmetry is used exclusively—however, there are descriptions in the superspace formalism available, see [GGRS83, chp. 5].

Given a d-dimensional space-time manifold $\mathcal{M}^d$ with Lorentz signature metric $g_{\mu\nu}$, a local (semi-)orthonormal basis of sections of the tangent bundle $\mathrm{T}\mathcal{M}^d \xrightarrow{\pi} \mathcal{M}^d$ can be chosen. In the language of physical literature, this amounts to introduce d vectors ${e_m}^\mu(x)$, such that

$${e_m}^\mu(x){e_n}^\nu(x)g_{\mu\nu}(x) = \eta_{mn}$$

holds for all $x \in \mathcal{M}^d$, where η_{mn} is the flat Minkowski metric. One also introduces corresponding inverse fields such that both

$${e_m}^\mu(x){e_\mu}^n(x) = \delta^n_m \qquad\text{and}\qquad {e_\mu}^m(x){e_m}^\nu(x) = \delta^\nu_\mu$$

are satisfied, such that conversely the space-time metric can be expressed as

$$g_{\mu\nu}(x) = {e_\mu}^m(x){e_\nu}^n(x)\eta_{mn}. \tag{6.6}$$

The fields ${e_\mu}^m(x)$, where $m = 0, \ldots, d-1$, are called **vielbein fields**, and in the view of (6.6) are sometimes referred to as the "square root of the metric". Using vielbein fields, there are

[d]The notion of a "chiral superfield" is often used in a more general context, where it is understood to be a certain irreducible superfield, see [GGRS83, §3.11]. Here the terms exclusively refers to the supermultiplets involving a chiral spin-$\frac12$ fermion.

now two sets of vector indices available: **world indices** $\mu, \nu, \rho, \dots$ and **local Lorentz indices** $m, n, l, \dots$, which are related via

$$V_m(x) = e_m{}^\mu(x) V_\mu(x) \qquad \text{and} \qquad V_\mu(x) = e_\mu{}^m(x) V_m(x). \tag{6.7}$$

It is important to realize that vielbein fields are not uniquely determined, as they have d^2 independent components in contrast to the $\frac{1}{2}d(d+1)$ components of the metric $g_{\mu\nu}$, where they are originating from. There is an invariance, which can be understood as a rotation that keeps η_{ab} invariant. This is called a **local Lorentz transformation**, and provided by

$$e'_m{}^\mu(x) = \Lambda_m{}^n(x) e_n{}^\mu(x) \qquad \text{and} \qquad e'_\mu{}^m(x) = e_\mu{}^n(x) \Lambda_n{}^m(x),$$

for $\Lambda_n{}^m$ being the components of the Lorentz group's standard representation. Since the Lorentz group's dimension is $\dim \mathcal{L}^d = \frac{1}{2}d(d-1)$, the number of independent components $d^2 - \frac{1}{2}d(d-1) = \frac{1}{2}d(d+1)$ equals the number of independent components found in the metric $g_{\mu\nu}$. The infinitesimal local Lorentz transformations are described in the same manner as ordinary Lorentz transformations, i.e. an antisymmetric tensor $\lambda^{ab} = -\lambda^{ba}$ is used to parameterize the generators.

In order to construct a local Lorentz invariant action involving spinors, a gauge field $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$ has to be introduced, which is called a **spin connection** under the discussed identification of (mathematical) connections with (physical) gauge fields. The spin connection $\omega_\mu{}^{ab}$ is related to the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$—which are not to be confused with the Γ-pairing components Γ^μ_{ab}—via

$$\partial_\mu e_\nu{}^m + \omega_\mu{}^m{}_n e_\nu{}^n = \Gamma^\lambda_{\mu\nu} e_\lambda{}^m,$$

and is completely determined by the vielbein fields, provided a certain condition to be torsion-free is satisfied, see [GSW87a, §4.3].

The field strength of a spin connection (i.e. the curvature of the connection) can be interpreted directly as the **space-time curvature**, which in turn is explicitly expressed via the Riemann curvature tensor

$$\begin{aligned} R_{\mu\nu}{}^{mn} &= \partial_\mu \omega_\nu{}^{mn} - \partial_\nu \omega_\mu{}^{mn} + \omega_\mu{}^m{}_l \omega_\nu{}^{ln} - \omega_\nu{}^m{}_l \omega_\mu{}^{ln} \\ &= \partial_\mu \omega_\nu{}^{mn} - \partial_\nu \omega_\mu^{mn} + [\omega_\mu, \omega_\nu]^{mn}, \end{aligned}$$

where the relations (6.7) are used to translate from the local Lorentz indices to the world indices. Obviously, the formal similarities to the Riemann curvature tensor

$$R_{\mu\nu}{}^\rho{}_\sigma = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\sigma},$$

when expressed in terms of the Christoffel symbols are striking. In essence, the vielbein fields provide another description of the metric by spanning a locally flat Minkowski space at any point of the underlying manifold.

6.7. Supergravity

The non-trivial brackets (6.4) of the odd SUSY generators induce translations on the space-time. Thus, upon localization of the supersymmetry by replacing $\epsilon^a \to \epsilon^a(x)$, local translations—better known as diffeomorphisms of the space-time—are induced. In other words, a theory invariant under local supersymmetry transformations is also invariant under general coordinate transformations. Since those naturally imply general relativity, local supersymmetry is usually referred to as **supergravity**.

In order to cancel the terms $\partial_\mu \epsilon^a$, which naturally arising upon localization in the kinetic terms of any physical theory involving supersymmetry, a gauge field $\chi^a_\mu(x)$ with one spinor and one vector index is introduced. This is the spin-$\frac{3}{2}$ **Rarita-Schwinger field**, which has a (local) SUSY transformation behavior

$$\delta_\epsilon \chi^a_\mu(x) \propto \partial_\mu \epsilon^a + \dots,$$

thus allowing for a cancellation of the $\partial_\mu \epsilon^a$-terms given the right coupling. The natural supersymmetric partner to the Rarita-Schwinger field is the space-time metric $g_{\mu\nu}$, which is described by the vielbein fields introduced in the previous section.[e] After quantization, the corresponding field quanta are identified as the spin-$\frac{3}{2}$ **gravitino** in the case of the Rarita-Schwinger field and the spin-2 **graviton** for the metric. Fortunately, the number of gravitino fields in a supergravity theory allows to easily read off the degree $\mathcal{N}$ of supersymmetry, which can be seen in the (later needed) SUGRA multiplets of tab. 6.2, 6.3 and 6.4.

The question of (non-)renormalizability of supergravity is a particularly involved issue. During the development of supergravity in the 70s, many scientists hoped that this could be the final theory quantum gravity—as supersymmetry seemed to counter the numerous divergences occuring in the quantization of gravity. However, it was soon found, that certain supergravities are non-renormalizable beyond a few loop orders, i.e. SUGRA is plagued by the same UV divergences any other field theory approach to quantum gravity suffers from. This led to the—perhaps premature—believe that all supergravities in general are non-renormalizable after a few loop orders. Recently, there are several hints that the maximal extended 4d $\mathcal{N} = 8$ SUGRA might indeed be finite[f] or at least better behaved regarding divergences:

4d	pure gravity	→	non-renormalizable to even 1 loop
	supergravity	→	renormalizable up to 2 loops, candidate divergent term in 3 loops—however, non-renormalizability is not proven
	max. SUGRA	→	hints to be renormalizable up to 4 loops, but perhaps not 5 loops—however, neither statement is proven
11d	supergravity	→	renormalizable up to 1 loop but not 2 loops.

Today SUGRA is usually conceived to be the effective field theory description of the massless particles arising in string theory. String theory in its current state is usually treated as the perturbative UV completion of supergravity.

[e] In principle, one could also consider a supermultiplet involving the spin-$\frac{3}{2}$ Rarita-Schwinger field and a spin-1 vector field. However, due to the general coordinate invariance (i.e. gravity) implied by localized supersymmetry, the spin-2 metric is naturally involved and by the Weinberg-Witten theorem can only form a supermultiplet with the spin-$\frac{3}{2}$ Rarita-Schwinger field. Thus, the SUGRA multiplet canned be avoided and other supermultiplets involving the Rarita-Schwinger field are not considered.

[f] There is a presentation [Dix07] by Dixon that summarizes the questions involved in those (non-)renormalizability issues and points to numerous references. In [GRV07] conditions on the non-renormalizability of maximal extended SUGRAs and certain SUGRA limits in the context of superstring theory are derived.

CHAPTER 7

Heterotic Strings

As the once promising theories of supergravity turned out to be not well-defined, another approach has to be taken to incorporate gravity into the established quantum framework, that allows to describe the other three known interactions. By replacing the point particle concept at the basis of quantum field theory with minimally extended objects, theories of 1-dimensional strings emerge. In this chapter the heterotic string will be introduced, both as a review and in order to fix the notation. First, the bosonic string and type-II superstring are constructed, which comprise the heterotic string. By now there is an abundance of literature covering those elementary subjects. The classic textbooks on superstring theory are the two volumes [GSW87a] and [GSW87b], but the material is now twenty years old and shows aging in some of the advanced sections. However, this is still the standard reference for questions regarding the anomaly cancellation in superstring theory or general results of rather technical nature. Polchinski's two volume text [Pol98a], [Pol98b] updates most of the material, introduces D-branes and covers the duality web of M-theory, which was recognized in 1995. Albeit [Joh03] focuses on D-branes, the introduction of string theory is carried out in detailed manner. Particularly noteworthy are the textbooks [DEF+99], which cover supersymmetry, quantum field theory and superstring theory from a more mathematical point of view. Recently, with the release of [BBS07], [Din07] and [Kir07], several up-to-date introductions to the subject became available.

7.1. The bosonic string

The motion of a point particle—just like any other physical object—is guided by the principle of least action. In the absence of exterior fields and interactions this translated to a minimization of the length of the world line, i.e. the line of the particle motion in space-time. The natural generalization to 1-dimensional extended objects of finite size—strings—is to minimize the area of the corresponding world surface.

In abstract terms the **string worldsheet** is described by a 2-dimensional surface Σ, i.e. a connected smooth manifold with $\dim_{\mathbb{R}} \Sigma = 2$ and local coordinates ξ^m for $m = 1, 2$. The **target space-time** $\mathcal{M} := \mathcal{M}_d := \mathcal{M}_{d-1,1}$ is a d-dimensional Lorentz manifold with "mostly plus" signature $(-+\cdots+)$, local coordinates x^μ for $\mu = 0, \ldots, d-1$ and local metric

$$\mathrm{d}s^2 = \langle \mathrm{d}x, \mathrm{d}x \rangle_G = G_{\mu\nu}(x)\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu.$$

Let $X : \Sigma \longrightarrow \mathcal{M}$ be a smooth function of the string worldsheet into the target space-time $\mathcal{M}$. The metric G on $\mathcal{M}$ induces a metric g on Σ via the pullback $g := X^*(G)$, which expands to

$$g = g_{mn}(\xi)\, \mathrm{d}\xi^m\, \mathrm{d}\xi^n, \qquad \text{where} \qquad g_{mn}(\xi) = G_{\mu\nu}\big(X(\xi)\big)\frac{\partial X^\mu}{\partial \xi^m}(\xi)\frac{\partial X^\nu}{\partial \xi^n}(\xi),$$

in terms of the chosen local coordinates on the worldsheet. The signature of this 2-dimensional metric can be of either Minkowskian $(-+)$ or Euclidean $(++)$ type. The local worldsheet coordinates are often chosen as (τ, σ) where σ refers to the string's spatial extension, such that $X(\tau, \sigma + 2\pi) = X(\tau, \sigma)$ describes a closed string for τ fixed. The **Nambu-Goto action** is then given by integrating over the 2-dimensional volume form dvol_g of the string worldsheet

$$S_{\mathrm{NG}}[X; G] = \int_\Sigma \mathrm{dvol}_g = \int_\Sigma \mathrm{d}^2\xi\, \sqrt{\det(g)}.$$

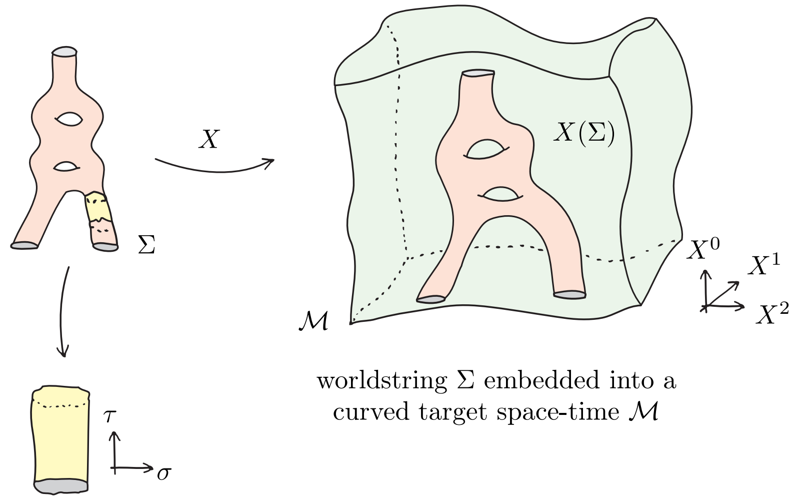

FIGURE 7.1. The bosonic degrees of freedom can be understood as an embedding of the worldsheet Σ into the target space-time.

Due to the non-linearity of the square root, the quantization of this action is difficult. Instead, one provides an artificial metric h for the string worldsheet Σ, which (in principle) is completely unrelated to G and g, and considers the **Polyakov action**

$$
\begin{aligned}
S_{\mathrm{P}}[X;h,G] &= \kappa \int_\Sigma \mathrm{dvol}_h \, \frac{1}{2} h^{mn} G_{\mu\nu}(X) \partial_m X^\mu \partial_n X^\nu \\
&= \kappa \int_\Sigma \mathrm{d}^2\xi \, \sqrt{\det(h)} \frac{1}{2} h^{mn} G_{\mu\nu}(X) \partial_m X^\mu \partial_n X^\nu,
\end{aligned}
\tag{7.1}
$$

where κ is a constant factor related to the string tension T, string length ℓ_{s} or Reggé slope α'. It can be shown (see [BBS07, ex. 2.6]) that the Polyakov action is classically equivalent to the Nambu-Goto action[a] and the equations of motion for the auxiliary metric can be used to eliminate h from the Polyakov action.

The Polyakov action can be understood from a much more general viewpoint as the action of a non-linear σ-model.[b] In the physical setting, the mapping $X : \Sigma \longrightarrow \mathcal{M}$ has a bosonic character, i.e. the component fields $X^\mu : \Sigma \longrightarrow \mathbb{R}$ are considered as d real (uncharged) bosonic scalars on the worldsheet. Certain additional terms can be added to the Polyakov action:

- The target space-time carries a 2-form $B \in \Omega^2_{\mathcal{M}}$, the so-called **antisymmetric tensor field** or **Kalb-Ramond field**. It can be added to the action by the term

  $$S_B = \kappa \int_\Sigma X^*(B)$$

  and implies from the physical point of view that the fundamental string of string theory is a source of the B-field much like charged particles are sources of the electromagnetic field.

[a] This is in fact a highly non-trivial issue: on one hand, the Polyakov action introduces an additional local symmetry (Weyl rescalings) and adds an additional field (artificial metric h). On the other hand, just as classical symmetries may have quantum anomalies, classical equivalent systems can show a dramatically different quantum behavior. In the case of bosonic string theory, however, the entire theory can be developed without any reference to the Polyakov action, as it is done in the undergraduate textbook [Zwi04]. The physical properties of the resulting (quantum) string theory are equal to those of the Polyakov approach.

[b] Non-linear σ-models are renormalizable when $\dim_{\mathbb{R}} \Sigma = 2$. Since this statement does not hold for $\dim_{\mathbb{R}} \Sigma > 2$, it is usually considered as a strong argument against the possible existence of fundamental membrane theories. [Fre99, lect. 4] is a very readable mathematical introduction to σ-models.

	count	particle	description
bosons (64b):	1	tachyon	causality-violating ground state
	1	graviton	symmetric tensor with fixed trace (35b)
	1	2-form	antisymmetric tensor (28b)
	1	dilaton	scalar (1b)

Table 7.1. Massless particle spectrum of the closed bosonic string.

- The target space-time may also carry a scalar **dilaton field** $\Phi \in \Omega^0_{\mathcal{M}} \cong C^\infty(\mathcal{M})$, which interacts with the string through the action term
$$S_\Phi = \kappa \int_\Sigma \mathrm{dvol}_h R_h \Phi(X),$$
where R_h is the Gaußian scalar curvature of the string worldsheet with respect to the metric h. This field is quite important for the (conceptual) calculation of scattering amplitudes as it is directly related to the topology of the string worldsheet by the Gauß-Bonnet theorem[c] and provides the coupling powers in a perturbative series, see [BBS07, §3.3].
- In principle, a physically undesirable **tachyon field** $T \in \Omega^0_{\mathcal{M}} \cong C^\infty(\mathcal{M})$ can be added using the action term
$$S_T = \kappa \int_\Sigma \mathrm{dvol}_h \, T(X).$$

The stationary points of the Polyakov action are the harmonic mappings $X : \Sigma \longrightarrow \mathcal{M}$, which gives a quite distinct appeal to the fundamental string equations. Furthermore, the action S_P is invariant under **Weyl rescalings** $h \mapsto \mathrm{e}^{2f} h$ of the string worldsheet metric for any function $f \in C^\infty(\Sigma)$, i.e. it constitutes a 2-dimensional conformal field theory on the string worldsheet. In short, the full bosonic string worldsheet action has the following properties:

(1) Invariance under orientation-preserving diffeomorphisms of the worldsheet Σ.
(2) Invariance under diffeomorphisms of the space-time $\mathcal{M}$.
(3) Renormalizability as a 2-dimensional conformal quantum field theory (non-linear σ-model) defined on the worldsheet Σ.
(4) Local dependence on the embedding X, the worldsheet metric h and the space-time metric G.

If attention is restricted to oriented 2-manifolds Σ, the oriented string worldsheet possesses a natural complex structure, i.e. it is in fact a Riemannian surface. The diffeomorphism invariance of the bosonic string action is a particular strong requirement and expressed through the **Virasoro algebra**, which will not be introduced here. The quantization of the particular conformal field theory requires $d = 26$ for anomaly cancellation in the Virasoro algebra and is responsible for the cancellation of nearly all negative norm states in the particle spectrum. This is usually referred to as the **supercritical** closed bosonic string. Details to all those issues are provided in the textbooks mentioned in the chapter's abstract.

After a Fourier mode expansion of the bosonic scalars $X^\mu : \Sigma \longrightarrow \mathbb{R}$ and the process of canonical quantization (see sec. 7.4 for the relevant cases), the corresponding mode operators fulfill the algebra of the raising and lowering ladder operators for the quantum-mechanical

[c] Recall from sec. 5.4 that the Gauß-Bonnet theorem direct relates the Riemann scalar curvature R with the Euler characteristic, given a closed (compact without boundary) 2-dimensional Riemannian manifold. Furthermore, since the Euler characteristic $\chi(\Sigma) = 2 - 2g(\Sigma)$ encodes the genus of a closed, orientable surface (recall sec. 3.6), the dilaton action term for closed strings can be interpreted as
$$\int_\Sigma R \, \mathrm{d}A = -4\pi\chi(\Sigma) = 8\pi\big(g(\Sigma) - 1\big).$$

harmonic oscillator. Since longitudinal excitations are not allowed by consistency, there remain $d-2=24$ independent directions transverse to the worldsheet, that can be excited. Different excitation states of those uncoupled oscillators yield different string states in the spectrum. This is the main paradigma of string theory: a single fundamental string represents a variety of different particle states by different modes of vibration.

A major difference between (perturbative) string theory and ordinary field theory is the description of interactions. A field theory is usually constructed from a number of free, non-interacting fields, where point-like interactions are introduced as a sort of "disturbances" of the free theory. This is also reflected in the Feynman diagrams: lines represent free particles, which only interact at certain (point-like) vertices. In contrast, string theory naturally contains interactions in the form of worldsheets not globally diffeomorphic to a cylinder (in case of closed strings). An interaction is simply the joining of several tubes, which by stretching (due to conformal invariance) is equivalent to a huge number of field theoretic interactions. This is again a strong indication for unified principles underlying the general string paradigma.

Finally, it remains to determine the massless particle spectrum of the closed bosonic string—massless, because even the lightest excited (massive) states have masses so far beyond experimental reach, that they might never be detected. Due to length considerations, the particle spectrum will not be derived in detail, as the general process to determine any string theory's (massless) particle spectrum is analogous to the considered case of the heterotic string in sec. 7.8. The massless particles described by the closed bosonic string are listed in tab. 7.1. Note the appearance of a tachyonic ground state, which renders the closed bosonic string entirely useless from the physical point of view. This will be dealt with in sec. 7.3 through the introduction of superstrings.

7.2. Background dependency of Polyakov formulation

A serious drawback of the current formulation of string theory is the dependence on the background fields of the target space-time. In the general physical bosonic string action

$$S_{\mathrm{bos}}[X;h,G,B,\Phi]=\kappa\int_\Sigma \mathrm{d}^2\xi\,\sqrt{\det(h)}\Bigg[R_h\Phi(X) + \left(\frac{1}{2}h^{mn}G_{\mu\nu}(X)+\frac{\epsilon^{mn}}{\sqrt{\det(h)}}B_{\mu\nu}(X)\right)\partial_m X^\mu\partial_n X^\nu\Bigg]$$

three background fields are found (with the unphysical tachyon field neglected): the 2-form field B, the dilaton field Φ and in particular the space-time metric G. Thus, the "stage" on which the strings propagate is already chosen and fixed, and string-interactions with graviton states only serve as perturbations around the fixed background space-time.[d]

From a conceptual viewpoint this is highly unsatisfactory, since space-time and strings are not handled in the same way. From a fundamental theory of quantum gravity one naturally would expect a full quantized treatment of the space-time itself. A background-independent formulation would requires the defining equations of the theory to be independent of the topology and shape of the space-time and the values of the various space-time fields. In particular, it must not refer to a specific space-time metric—as the full bosonic string action clearly does. Rather, the different backgrounds or configurations should be obtained as different solutions of the underlying equations.

In the context of string theory it is hoped, that the current background dependent formulation can be replaced by a non-perturbative background independent description, particularly in the context of M-theory. A general account on background (in)dependence is found in

[d] In principle, this is not surprising, as the current formulation of string theory is purely perturbative, such that a fixed background stage is actually required to calculate the different worldsheets to all loop orders. Doing so without a reference metric would be on equal footage to a Taylor expansion without a fixed expansion point.

[RCD+05]. In [Wit93] a detailed account on the background dependence problem of string theory is presented. Recent results (that far surpass the aims of this exposition) show that space-time is indeed an emergent concept in string theory, but much work remains to be done in that direction.

7.3. Supersymmetric strings

The main drawback of the bosonic string is the lack of any fermionic excitation states and the presence of a causality-violating tachyon in the particle spectrum, cf. fig. 7.2. The solution to both problems is the introduction of fermions—more precisely, a 1-dimensional (minimal) fermionic degree of freedom for each bosonic one—which are in supersymmetric relation to the bosonic degrees of freedom. This can be achieved along different roads:

- *Ramond-Neveu-Schwarz (RNS) formulation:* In this approach one introduces worldsheet fermions on Σ, which at first only yield $\mathrm{SO}(1,d-1)$ space-time vectors ψ^μ in addition to the bosonic space-time vectors X^μ. For free closed superstrings there are two different spin structures, equivalent to periodic and anti-periodic boundary conditions, called Ramond and Neveu-Schwarz due to historical reasons. The GSO projection then discards problematic states and yields a space-time supersymmetric spectrum of states, and in particular space-time fermions.
- *Green-Schwarz (GS) formulation:* Conceptually this approach is much more natural than the Ramond-Neveu-Schwarz construction. Using the superspace formalism developed in the last chapter, one simply adds fermionic space-time spinors θ^α to the bosonic fields X^μ and automatically gains a supersymmetric particle spectrum. It is, however, much more complicated to do actual calculations in the superspace formulation of string theory and the manifestly covariant quantization is rather complicated.

Due to its simplicity, the RNS superstring construction will be investigated in the following. An extensive introduction to the GS construction is found in [DEF+99, II-Strings, lect. 10] or [BBS07, chp. 5].

The basic idea is to add d worldsheet spinors of well-defined chirality in correspondence to the d worldsheet bosons, which at first behave like a space-time $\mathrm{SO}(1,d-1)$-vector ψ^μ. The spinors are introduced as follows, where now the mathematical existence and uniqueness results of chap. 4 are used:

- A *Minkowskian signature* worldsheet admits Majorana-Weyl spinors, i.e. real spinors with defined chirality, such that a Weyl spinor ψ^μ can be split in two Majorana-Weyl spinors ψ^μ_+ and ψ^μ_-. Those are section of the "square root" spinor bundle[e] $S = K^{\frac{1}{2}} \xrightarrow{\tilde{\pi}} \Sigma$ associated to the canonical line bundle $K_\Sigma = \Lambda^2 \mathrm{T}^*\Sigma$, which exist if $w_2(K) = 0$, cf. p. 40, and the different spin structures are labeled by $H^1(\Sigma;\mathbb{Z}_2)$.
- For an *Euclidean signature* worldsheet, there are no Majorana-Weyl spinors, but one can circumvent this problem by usage of a single complex Weyl spinor, i.e. ψ^μ_+ is a section of the complex spin bundle $K^{\frac{1}{2}} \xrightarrow{\pi} \Sigma$ and $\psi^\mu_- \in \Gamma(\bar{K}^{\frac{1}{2}})$ is the complex conjugate of ψ^μ_+.

As detailed on p. 30, the vanishing of the second Stiefel-Whitney class w_2 for complex vector bundles is implied by $c_1(K) = 0 \bmod 2$, and due to

$$c_1(K) = -\chi(\Sigma) = 2g(\Sigma) - 2 = 2\big(g(\Sigma) - 1\big)$$

this requirement is met by any worldsheet Σ. For a closed free string, the worldsheet takes the form of an open cylinder $I \times S^1$, where I is an open interval. Due to $H^1(I \times S^1;\mathbb{Z}_2) \cong \mathbb{Z}_2$

[e] A **square root bundle** is another approach to capture the concept of 2-fold coverings, which is generic to spinors. It relies on the fact, that any non-zero complex number $z \in \mathbb{C}^\times$ has exactly two square roots $\pm\sqrt{z}$. See [LM89, app. D] for a more detailed introduction.

there are exactly two possible spin structures over a free superstring. If $(\tau,\sigma)\sim(\tau,\sigma+2\pi)$ are the coordinates of $I\times S^1$, the different spin structures can be specified as follows:[f]

$$
\begin{aligned}
&\psi^\mu_\pm(\tau,\sigma+2\pi) = +\psi^\mu_\pm(\tau,\sigma) && \textbf{Ramond}\text{ (R): periodic on cylinder}\\
&\psi^\mu_\pm(\tau,\sigma+2\pi) = -\psi^\mu_\pm(\tau,\sigma) && \textbf{Neveu-Schwarz}\text{ (NS): anti-periodic on cylinder}
\end{aligned}
\tag{7.2}
$$

The RNS fermion fields $\psi^\mu_\pm$ give rise to additional negative norm states, which are not eliminated by the Virasoro constraints of the bosonic string. Since the fields $\psi^\mu_\pm$ are of fermionic nature, matching local constraints can only be formulated by the usage of anticommuting quantities. Such constraints are local supersymmetries, i.e. **supergravities**, on the worldsheet Σ. To provide an supersymmetric partner particle to the spin-2 graviton, the gravitino field χ^ς_m of spin-$\frac{3}{2}$ has to be introduced, where m is the worldsheet vector index and $\varsigma=\pm$ the worldsheet spinor index. As before, to introduce spinors in an arbitrary background, the vielbein formalism is used, i.e. a local zweibein frame ${e_m}^a$, where $a=1,2$ are the local Lorentz indices and m-indices refer to worldsheet coordinate vectors. The O(2)-invariant frame metric is the Euclidean metric δ_{ab}, and the inverse frame is denoted ${e_a}^m$, i.e. there are the relations

$$
{e_a}^m{e_m}^b = \delta^b_a, \qquad h_{mn} = {e_m}^a{e_n}^b\delta_{ab},
$$

as introduced in sec. 6.6 of the previous chapter. Furthermore, let ρ^m denote the canonical representation of the 2-dimensional Clifford algebra, i.e. 2×2-matrices satisfying $\{\rho^a,\rho^b\} = -2\delta^{ab}$. The first $\mathcal{N}=1$ supergravity action for the situation here was constructed in [DZ76] and [BDH76]:

$$
\begin{aligned}
S_{\mathrm{II}}[X,\psi;h,G,\chi] = \kappa\int_\Sigma \mathrm{dvol}_h\Big(&\frac{1}{2}h^{mn}\partial_m X^\mu\partial_n X^\nu - \mathrm{i}\bar\psi^\mu\rho^a\overbrace{{e_a}^m\partial_m}^{\nabla_a}\psi^\nu\\
&+2\bar\chi_a\rho^b\rho^a\psi^\mu {e_b}^m\partial_m X^\nu + \frac{1}{2}\bar\psi^\mu\psi^\nu\bar\chi_m\rho^n\rho^m\chi_n\Big)G_{\mu\nu}(X).
\end{aligned}
\tag{7.3}
$$

This action only makes sense if the spin structures of both the worldsheet spinors ψ^μ and gravitino spinor-vectors χ^ς_m are the same. The first term is just the Polyakov action S_{bos} introduced in the sec. 7.1. In the second term, the free spinor fields are described, whereas the third and fourth term arise due to the requirement of local supersymmetry. The symmetries of the type-II superstring action can be summarized as follows:

(1) Invariance under orientation-preserving diffeomorphisms of the worldsheet.
(2) Space-time Poincaré-invariance.
(3) Local 2-dimensional supersymmetry of the worldsheet:

$$
\begin{cases}
\delta_\epsilon X^\mu = \bar\epsilon\psi^\mu\\
\delta_\epsilon\psi^\mu = -\mathrm{i}\rho^m\epsilon(\partial_m X^\mu - \bar\psi^\mu\chi_m)\\
\delta_\epsilon {e_m}^a = -2\mathrm{i}\bar\epsilon\rho^a\chi_m\\
\delta_\epsilon\chi_m = \nabla_m\epsilon
\end{cases}
$$

(4) Weyl-invariance of the worldsheet:

$$
\begin{cases}
\delta_\omega X^\mu = 0\\
\delta_\omega\psi^\mu = -\frac{1}{2}\omega\psi^\mu\\
\delta_\omega {e_m}^a = \omega {e_m}^a\\
\delta_\omega\chi_m = \frac{1}{2}\omega\chi_m
\end{cases}
$$

(5) Super-Weyl-invariance of the worldsheet:

$$
\begin{cases}
\delta_\eta X^\mu = \delta_\eta\psi^\mu = \delta_\eta {e_m}^a = 0\\
\delta_\eta\chi_m = \mathrm{i}\rho_m\eta
\end{cases}
$$

[f] Often the coordinates (τ,σ) are composed into a single complex coordinate $w := \tau+\mathrm{i}\sigma$ and the cylinder is conformally mapped to an annulus in $\mathbb{C}$ by $w\mapsto \mathrm{e}^w$. In terms of the annulus coordinate z, the Ramond-periodic and Neveu-Schwarz-antiperiodic boundary conditions are reversed, see [DEF+99, p. 918].

(6) Local 2-dimensional Lorentz-invariance of the worldsheet, expressed by local O(2) frame rotations.

The numerous symmetries are imposed as super-Virasoro constraints in the quantizations process. Similar to the Virasoro anomaly found in the quantization of the bosonic string, the super-Virasoro constraints of the superstring also yield an anomaly which only cancels in $d = 10$ dimensions. Again, this is called the **supercritical** case of the closed superstring theory.

7.4. Equations of motion and quantization

The quantization of the bosonic string and superstring is difficult due to the numerous symmetries of the action. There are several different approaches to deal with this problem:

- In the canonical quantization the classical string variables are replaced by matching operators just like in the quantization process of classical fields. Due to the constraints of the (super-)string system, there are two essential ways in this direction:
  - In the old **covariant canonical approach** the unconstrained classical string variables are canonically quantized and the constraints of the system are imposed in the quantum theory as conditions on the states, i.e. the classical constraints are turned into restrictions on the Hilbert space of states.
  - Alternatively, one can already solve the constraints at the classical level and quantize afterwards. To actually solve the constraints, a light-cone gauge is used. However, in this **light-cone gauge quantization** Lorentz invariance is lost at first, and it requires considerable work afterwards to check that it is still a symmetry of the quantized system.
- The **path integral** or **BRST quantization** technique is conceptually quite similar to the path integral quantization of classical fields. One introduces the Faddeev-Popov ghost fields known from classical gauge theory in order to achieve a manifest Lorentz invariant quantization.[g]
- Using pure spinors, Berkovitz introduced a method of **supercovariant quantization** of the superstring directly in the GS formalism, that manifestly keeps supersymmetry and Lorentz invariance. However, this approach is not yet found in any textbook, see [Ber02].

In the following considerations the light-cone gauge quantization will be used. Using the reparametrization invariance of the worldsheet, two of the three independent components of the worldsheet metric h—which is a symmetric tensor—can be chosen and by Weyl rescalings the remaining one can also be fixed, such that

$$h_{mn} = \eta_{mn} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

becomes the flat 2-dimensional Minkowski metric.[h] The local supersymmetry and super-Weyl symmetry of the full superstring action (7.3) can be used to locally set the four components of the gravitino vector-spinor $\chi_m = 0$, i.e. one is left with the global supersymmetric action, that

[g] In fact, the BRST quantization has many appealing mathematical properties: A fundamental property of the BRST charge operator Q_{BRST} is its nilpotency, i.e. ${Q_{\mathrm{BRST}}}^2 = 0$. In a certain context, this operator can be regarded as a generalized cochain mapping and the cohomology classes of the resulting (generalized) cohomology theory correspond to the physical states of the quantized string theory. A short mathematical introduction to BRST cohomology with a general outlook to its physical applications is found in [Fig06a], whereas [BBS07, p. 78f] explains the general line of reasoning involved in this construction.

[h] Note that in general there are topological obstructions, i.e. the Euler characteristic $\chi(\Sigma)$ has to vanish for a globally flat choice of the worldsheet metric h. Nevertheless, it is quite notable that in the case of 2d string worldsheets there is exactly enough gauge symmetry such that the artificially introduced worldsheet metric can be completely fixed.

essentially consists of the first two terms of (7.3). Using the Hamiltonian principle of least action, one deduces that the **equations of motion for the bosonic fields** are

$$\Box_h^2 X^\mu(\tau,\sigma) = \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu(\tau,\sigma) = 0$$

with the **closed-string boundary condition** $X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)$. In order to solve the bosonic equations—which are in fact the ordinary 1-dimensional wave equations—one splits into left- and right-moving parts, i.e. $X^\mu(\tau,\sigma) = X^\mu_{\mathrm{L}}(\tau,\sigma) + X^\mu_{\mathrm{R}}(\tau,\sigma)$. Respecting the closed string boundary conditions $X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)$, the mode expansion are

$$\begin{aligned} X^\mu_{\mathrm{L}}(\tau,\sigma) &= \frac{1}{2}x_0^\mu + \frac{1}{2}p^\mu_{\mathrm{L}}(\tau+\sigma) + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_{\mathrm{L},n}\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)} \\ X^\mu_{\mathrm{R}}(\tau,\sigma) &= \frac{1}{2}x_0^\mu + \frac{1}{2}p^\mu_{\mathrm{R}}(\tau-\sigma) + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_{\mathrm{R},n}\mathrm{e}^{-\mathrm{i}n(\tau-\sigma)}, \end{aligned} \tag{7.4}$$

where the reality conditions $(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$ hold for both the L- and R-movers. In particular, the left- and right-moving momenta p^μ_{L} and p^μ_{R} are equal. For the splitting $\alpha^\mu_n = a^\mu_n - \mathrm{i}b^\mu_n$, where a^μ_n and b^μ_n are real coefficients, the reality condition implies $\alpha^\mu_{-n} = a^\mu_n + \mathrm{i}b^\mu_n$, such that using the summation

$$\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}\mathrm{e}^{-\mathrm{i}n(\tau\pm\sigma)} = -2\mathrm{i}\sum_{n>0}\frac{1}{n}\Big[a^\mu_n \sin\big(n(\tau\pm\sigma)\big) + b^\mu_n\cos\big(n(\tau\pm\sigma)\big)\Big]$$

the mode expansions of the closed bosonic string can be directly formulated in terms of real Fourier series

$$\begin{aligned} X^\mu_{\mathrm{L}}(\tau,\sigma) &= \frac{1}{2}x_0^\mu + \frac{1}{2}p^\mu_{\mathrm{L}}(\tau+\sigma) + \sum_{n>0}\frac{1}{2}\Big[a^\mu_{\mathrm{L},-1}\sin\big(n(\tau+\sigma)\big) + b^\mu_{\mathrm{L},-1}\cos\big(n(\tau+\sigma)\big)\Big] \\ X^\mu_{\mathrm{R}}(\tau,\sigma) &= \frac{1}{2}x_0^\mu + \frac{1}{2}p^\mu_{\mathrm{R}}(\tau-\sigma) + \sum_{n>0}\frac{1}{2}\Big[a^\mu_{\mathrm{R},-1}\sin\big(n(\tau-\sigma)\big) + b^\mu_{\mathrm{R},-1}\cos\big(n(\tau-\sigma)\big)\Big]. \end{aligned}$$

However, for convenience the "complex" mode expansions (7.4) will be used in most cases.

Along similar lines of reasoning one derives the **equations of motion for the fermionic fields** on the 2-dimensional string worldsheet:

$$\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma}\right)\psi^\mu_+(\tau,\sigma) = 0 \qquad \text{and} \qquad \left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma}\right)\psi^\mu_-(\tau,\sigma) = 0$$

Depending on the type of spin structure used for the fermionic fields $\psi^\mu_\pm$, there are different mode expansions for the Ramond and Neveu-Schwarz boundary conditions:

$$\underbrace{\begin{aligned} \psi^\mu_+(\tau,\sigma) &= \sum_{n\in\mathbb{Z}} d^\mu_{+,n}\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)} \\ \psi^\mu_-(\tau,\sigma) &= \sum_{n\in\mathbb{Z}} d^\mu_{-,n}\mathrm{e}^{-\mathrm{i}n(\tau-\sigma)} \end{aligned}}_{\text{Ramond (R)}} \qquad \text{and} \qquad \underbrace{\begin{aligned} \psi^\mu_+(\tau,\sigma) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_{+,r}\mathrm{e}^{-\mathrm{i}r(\tau+\sigma)} \\ \psi^\mu_-(\tau,\sigma) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_{-,r}\mathrm{e}^{-\mathrm{i}r(\tau-\sigma)} \end{aligned}}_{\text{Neveu-Schwarz (NS)}} \tag{7.5}$$

Finally, the canonical quantization of the 2-dimensional worldsheet field theory is carried out by imposing the (anti-)commutation relations

$$\begin{array}{lll} \text{bosonic:} & \big[\alpha^\mu_{\mathrm{L},m},\alpha^\nu_{\mathrm{L},n}\big] = \big[\alpha^\mu_{\mathrm{R},m},\alpha^\nu_{\mathrm{R},n}\big] = m\delta_{m,-n}\eta^{\mu\nu} & \\ & \big[x^\mu_0, p^\mu_0\big] = \mathrm{i}\eta^{\mu\nu} & \\ \text{fermionic:} & \big\{d^\mu_{+,m}, d^\nu_{+,n}\big\} = \big\{d^\mu_{-,m}, d^\nu_{-,n}\big\} = \eta^{\mu\nu}\delta_{m,-n} & \text{(Ramond)} \\ & \big\{b^\mu_{+,r}, b^\nu_{+,s}\big\} = \big\{b^\mu_{-,r}, b^\nu_{-,s}\big\} = \eta^{\mu\nu}\delta_{r,-s} & \text{(Neveu-Schwarz)} \end{array}$$

on the modes. Due to the obvious sign similarity in the mode expansions of ψ_+^μ and X_{L}^μ or ψ_-^μ and X_{R}^μ, the ψ_+^μ-spinor is regarded as L-moving, whereas ψ_-^μ is a R-mover. In the following, the worldsheet index $\varsigma = \pm$ will be replaced accordingly. The underlying independence of the right- and left-moving parts of the bosonic and fermionic string degrees of freedom will be crucial for the construction of the heterotic string in section sec. 7.6.

Note that the choice of fermion boundary conditions also applies to the gravitino vector-spinor χ_m^ς, which yields four possible combinations considering the left-/right-mover splitting: R-R, NS-NS, R-NS, NS-R. By means of the GSO projection (see [DEF+99, II-Strings, §7.7] and sec. 7.7) this essentially reduces to a choice of chirality for the L- and R-moving ground state, which either leads to non-chiral **type-IIA** or chiral **type-IIB superstring theory**. For open strings, a similar **type-I superstring theory** can be constructed. However, either type of superstring theory for itself is also a highly unsatisfactory construction comparable to the bosonic string, cf. fig. 7.2. This will be dealt with by introducing the heterotic string in sec. 7.6, after a short digression on a necessary tool to construct it.

7.5. Toroidal compactification of closed bosonic strings

In order to deal with the large number of dimensions ($d = 26$ in the case of bosonic strings and $d = 10$ for superstrings), one approach—going back to the early 20th century work of Kaluza and Klein, who tried to unify electrodynamics and general relativity—is to compactify some of the undesired spatial dimensions on a higher dimensional torus. By providing n linearly independent vectors $e_i \in \mathbb{R}^n$ for $i = 1, \dots, n$, a **lattice**

$$\Lambda := \left\{ \sum_{i=1}^n a_i e_i \text{ for } a_i \in \mathbb{Z} \right\} = \sum_{i=1}^n \mathbb{Z} e_i$$

is defined via integer multiples, such that the geometry of a n-torus $T_\Lambda^n := \mathbb{R}^n/\Lambda$ is completely specified. The action of Λ on $\mathbb{R}^n$ is to be understood in the translational sense, i.e. two points $x, y \in \mathbb{R}^n$ are identified if the difference $x - y$ is on the lattice Λ. A n-torus constructed in such a fashion and with local coordinates $(y^1, \dots, y^n)$ has the natural metric

$$\mathrm{d}s^2 = \sum_{i,j=1}^n G_{ij}\, dy^i\, \mathrm{d}y^j,$$

where the metric components are defined by $G_{ij} := \langle e_i, e_j \rangle$. In particular, for an orthonormal lattice—a torus where all internal circles are perpendicular and of unit size—this torus metric reduces to $G_{ij} = \delta_{ij}$.

In general, given a lattice Λ defined as points in n-dimensional vector space an inner product V, there is a corresponding **dual lattice** $\Lambda^\star$ induced by the inner product of V, i.e.

$$\Lambda^\star := \{ v \in V \text{ such that } v \cdot w \in \mathbb{Z} \text{ for all } w \in \Lambda \}.$$

Since $\Lambda^\star$ is obviously a lattice again, a set of dual basis vectors $\{e_i^\star\}$ can be chosen, such that

$$\Lambda^\star = \left\{ \sum_{i=1}^n b_i e_i^\star \text{ for } b_i \in \mathbb{Z} \right\} \qquad \text{and} \qquad e_i^\star \cdot e_j = \delta_{ij}$$

are satisfied. If $\Lambda = \Lambda^\star$ holds, the lattice is called **self-dual**. Furthermore, a lattice is called **integral** if $v \cdot w \in \mathbb{Z}$ for all $v, w \in \Lambda$ and **even** if Λ is both integral as well as $v^2 \in 2\mathbb{Z}$ for all $v \in \Lambda$. Further details on lattices in the context of string theory are found in [GO84].

The closed string has to satisfy certain conditions to consistently propagate on the toroidal compactified space-time $\mathcal{M}_{d-n} \times T_\Lambda^n$, which naturally inherits the product metric

$$\mathrm{d}s^2 = \mathrm{d}s^2_{\mathcal{M}_{d-n}} + \mathrm{d}s^2_{T_\Lambda^n} = \sum_{\mu,\nu=0}^{d-n-1} \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu + \sum_{i,j=1}^n G_{ij}\, dy^i\, \mathrm{d}y^j,$$

The bosonic string is still well-defined on this space-time, if the modified boundary conditions

$$\begin{aligned} X^\mu(\tau,\sigma+2\pi) &= X^\mu(\tau,\sigma) && \text{for flat dimensions } \mu = 0,\dots,d-n-1 \\ X^i(\tau,\sigma+2\pi) &= X^i(\tau,\sigma) + 2\pi W^i && \text{for compactified dimensions } i = 1,\dots,n \end{aligned}$$

are satisfied. The additional term $2\pi W^i$ describes windings around the toroidal compactified dimensions and W is called the **winding vector**. Of course, this modifies the mode expansion (7.4) of the compactified coordinates such that the left- and right-moving momenta p^i_{R} and p^i_{L} are no longer equal. Rather, their difference

$$p^i_{\mathrm{L}} - p^i_{\mathrm{R}} = 2W^i$$

is related to the winding of the string and becomes discrete, as $W^i \in \mathbb{Z}$. In this context, the compactified internal momenta are called the **Kaluza-Klein excitations** $K_i \in \mathbb{Z}$, such that

$$p^i_{\mathrm{L}} + p^i_{\mathrm{R}} = K^i$$

holds in the simplest case. In the possible presence of the antisymmetric background 2-form B-field or a non-orthogonal torus metric G_{ij}, this expression for the Kaluza-Klein excitations is modified (see [BBS07, §7.3]), such that the left- and right-moving parts of the momenta are given by

$$\begin{aligned} p^i_{\mathrm{L}} &= W^i + G^{ij}\left(\frac{1}{2}K_j - B_{jk}W^k\right) \\ p^i_{\mathrm{R}} &= -W^i + G^{ij}\left(\frac{1}{2}K_j - B_{jk}W^k\right). \end{aligned}$$

An important consistency condition is (one-loop) modular invariance (see [Pol98a, chp. 7], [GSW87b, chp. 8] and in particular [BBS07, p. 274] for details): In the calculation of one-loop string amplitudes, one has to integrate over all possible genus-one world sheets. The geometry of such torus worldsheets is fully specified by a single complex parameter ϱ, which is unfortunately not unique—several values of ϱ, which are related by modular transformations, describe the same one-loop worldsheet torus. To avoid additional contributions due to multi-counting of the same geometry, the integration over the the parameter ϱ is restricted to a certain fundamental region. **One-loop modular invariance** guarantees that the amplitude function inside the integral is invariant under modular transformations.

bosonic string:

con:
- no fermions
- tachyonic state
- no closed-string Yang-Mills states

superstring:

pro:
- fermions
- local supersymmetry
- tachyon-free

con:
- no closed-string Yang-Mills states
- not unique (type I, IIA, IIB)

heterotic string:

pro:
- fermions
- local supersymmetry
- tachyon-free
- closed-string Yang-Mills states
- unique up to choice of gauge group

FIGURE 7.2. The unification of the unsatisfactory bosonic string and superstring theory in the heterotic string avoids the problems of both approaches.

One-loop modular invariance requires $p_{\mathrm{L}}^2 - p_{\mathrm{R}}^2 = -2W^i K_i$, which is automatically guaranteed by the above equations. Furthermore, the left- and right-moving momenta of the toroidal compactified momenta must live on an even, self-dual lattice $\Lambda_{n,n}$ with signature $\big((+1)^n, (-1)^n\big)$, where

$$p = \big(p_{\mathrm{L}}^i, p_{\mathrm{R}}^i\big)$$

are the lattice vectors. Obviously, this choice of signature reproduces $p^2 = p_{\mathrm{L}}^2 - p_{\mathrm{R}}^2$. The self-duality requirement comes from the **T-duality** (or rather the generalized $\mathbf{O}(\boldsymbol{n},\boldsymbol{n};\mathbb{Z})$**-duality**, see [BBS07, p. 269]) of the toroidal compactified dimensions, which in the case of an orthogonal torus inverts the compactification radii and exchanges windings and Kaluza-Klein excitations.

Naturally, the mathematical question arises what kind of even self-dual lattices exist for a given n, or more generally for signature (n_1, n_2). In the study of modular forms, cf. [Ser73, §VII.6], one finds that only lattices with signature

$$n_1 - n_2 \equiv 0 \bmod 8$$

can be even and self-dual. Furthermore, the toroidal compactification has another virtue in the general context of Kaluza-Klein Yang-Mills states, which will be reviewed in chap. 8.

7.6. The heterotic string

The original shortcomings of the bosonic string, i.e. the lack of fermions and the tachyonic state were dealt with by the introduction of the superstring. Moreover, the bosonic string only possesses Yang-Mills gauge states in the case of open strings, where Chan-Paton charges are assigned to the endpoint. Thus, a theory of closed strings that contains Yang-Mills gauge states would be desirable. In 1984 a rather odd idea was put forward, cf. [GHMR85a]: adjoin the left-moving part of a bosonic string in 26-dimensional space-time with the right-moving part of type-II superstring in 10-dimensional space-time. The obvious difference in space-time dimensions has to be accounted for by toroidally compactifying the 16 additional dimensions of the left-moving bosonic string. By the Kaluza-Klein mechanism, which will be investigated in detail in the next chapter, this (partial) compactification provides the closed string Yang-Mills states one is missing in both the closed bosonic string and type-II superstring, see [BL94, §9] in this context. It is quite noteworthy, that the Euclidean signature $(16, 0)$ of the required lattice can be even and self-dual by the above condition.

The classification of even self-dual lattices of such signature implies, that (up to isomorphisms) there are just two different types of lattices, which naturally arise in the representation theory of certain 496-dimensional Lie groups: either the root lattice $\Lambda_{\mathrm{E}_8 \times \mathrm{E}_8} \cong \Lambda_{\mathrm{E}_8} \times \Lambda_{\mathrm{E}_8}$ of the exceptional group $\mathrm{E}_8 \times \mathrm{E}_8$ or the root lattice $\Lambda_{\mathrm{SO}(32)}$ of the group SO(32), both of which are given in app. B. This construction yields the heterotic closed string in 10-dimensional space-time with either $\mathrm{E}_8 \times \mathrm{E}_8$ or SO(32) Yang-Mills gauge states.[i] In the context of the heterotic string, the 26-dimensional component index $\mu = 0, \dots, 25$ of the fields is usually renamed in the way indicated in fig. 7.3, that is:

$\mu = \pm$	2 coordinates tangential to the worldsheet, e.g. $X^+ := \frac{1}{\sqrt{2}}\left(X^0 + X^3\right) \qquad X^- := \frac{1}{\sqrt{2}}\left(X^0 - X^3\right)$
$\mu = 1, 2$	2 transverse coordinates in flat 4d space-time
$i = 1, \dots, 6$	6 coordinates which have to be compactified
$\big(\mu = 1, \dots, 8$	8 non-tangential coordinates, i.e. including the i-labeled coordinates $\big)$
$I = 1, \dots, 16$	16 internal gauge degrees of freedom

[i]See sec. 7.8 for an alternative construction of the heterotic string's gauge degrees of freedom, which leads to the same conclusion base on a choice of certain fermionic boundary conditions / spin structures.

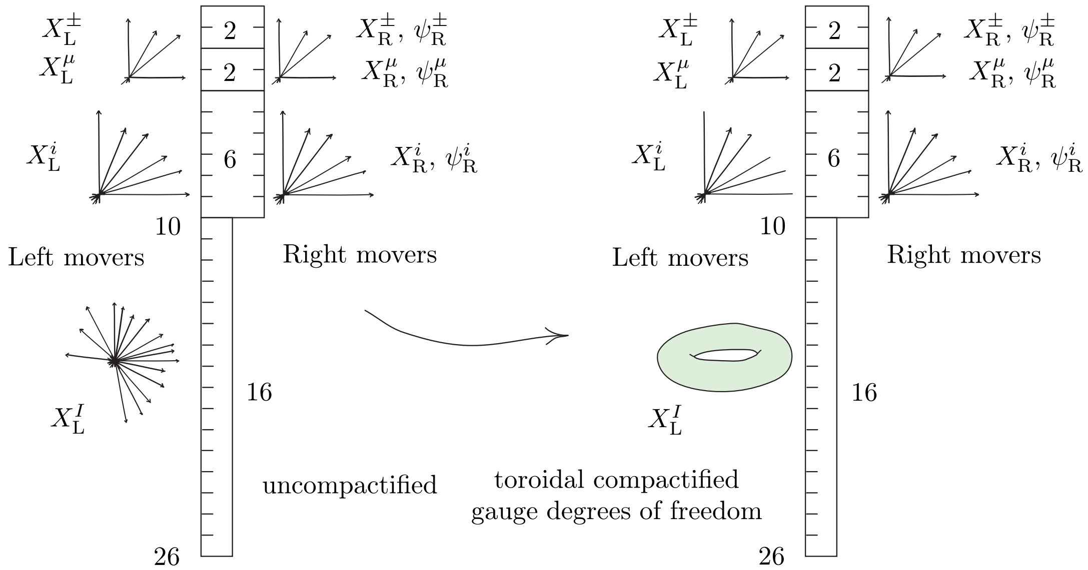

FIGURE 7.3. The image shows the distribution and naming of the 26 dimensions of the left-moving (supercritical) bosonic string, which are partially matched by 10 dimensions of the right-moving (supercritical) type-II superstring in the heterotic string. Using a suitable choice of coordinates, the 10 matched dimensions can be split into 6 coordinates X^i that are about to be compactified and 4 dimensions, which are further separated into 2 coordinates $X^\pm$ tangential and 2 coordinates X^μ transverse to the worldsheet. The 16 additional unmatched components of the left-moving bosonic string are toroidal compactified, yielding a theory of closed strings with internal $E_8 \times E_8$ or SO(32) Yang-Mills gauge states.

The internal gauge degrees of freedom are the remaining "bosons" of the left-moving bosonic string and accordingly it is called the **bosonic construction** of the heterotic string. There is an equivalent construction in terms of 32 Majorana-Weyl fermions, called the **fermionic construction**. However, in both cases those are not physical bosons or fermions, as there are no matching states in the R-moving part.

Miraculously, it turned out, that this rather odd construction is in fact anomaly free (see [GS84] and [GS85]), and was further developed in [GHMR85b] and [GHMR86]. This discovery inaugurated the "first superstring revolution", which brought string theory to a much wider audience. One might ask why the toroidal compactification is only carried out for the 16 unmatched left-moving bosonic coordinates and not further down to just 4 remaining flat dimensions by usage of a (22,6) instead of a (16,0) lattice. In fact, it will be shown in sec. 8.4, that such a compactification leads directly to highly unrealistic, $\mathcal{N} = 4$ extended supersymmetry (which implies no chiral matter fermions) in the effective 4d theory.

7.7. Ground states and GSO projection

The particle spectrum in string theory arises from the different types of excitations of the harmonic oscillators found in the string's degrees of freedom. Only transverse oscillations of the string are allowed, which effectively removes two degrees of freedom for excitations. With respect to general coordinates (i.e. without special reference to a coordinate system, where the

24 or 8 transverse coordinates are singled out) the **excitation numbers** are defined by

$$N_{\mathrm{L,bos}} := \sum_{n=1}^{\infty}\sum_{\mu=0}^{25} \alpha^{\mu}_{\mathrm{L},-n}\alpha^{\mu}_{\mathrm{L},n} \qquad N_{\mathrm{R,bos}} := \sum_{n=1}^{\infty}\sum_{\mu=0}^{9} \alpha^{\mu}_{\mathrm{R},-n}\alpha^{\mu}_{\mathrm{R},n}$$

$$\text{(Ramond)} \qquad N^{\mathrm{R}}_{\mathrm{R,ferm}} := \sum_{n=1}^{\infty}\sum_{\mu=0}^{9} n d^{\mu}_{\mathrm{R},-n} d^{\mu}_{\mathrm{R},n}$$

$$\text{(Neveu-Schwarz)} \qquad N^{\mathrm{NS}}_{\mathrm{R,ferm}} := \sum_{r=\frac{1}{2}}^{\infty}\sum_{\mu=0}^{9} r b^{\mu}_{\mathrm{R},-r} b^{\mu}_{\mathrm{R},r}$$

which count the number of oscillators acting on the ground states. In the RNS-construction of the right-moving superstring the ground states are described as follows:

The right-moving **Neveu-Schwarz ground state** $|0;\mathrm{NS}\rangle_{\mathrm{R}}$ transforms as a space-time scalar under $\mathrm{SO}(1,9)$ and is thus a bosonic state. The entire Neveu-Schwarz sector is obtained by applying raising operators b^{μ}_r for $r<0$ to this ground state, and all obtained states transform under the respective tensor representations of $\mathrm{SO}(1,9)$. Thus, the entire Neveu-Schwarz sector consists solely of space-time bosons.

On the other hand, the right-moving **Ramond ground state** $|0,\alpha;\mathrm{R}\rangle_{\mathrm{R}}$ is degenerate (indicated by the index α) and transforms as a massless $\mathrm{Spin}(1,9)$ Dirac spinor, see[DEF⁺99, II-Strings, §§7.2,7.2]. This becomes obvious from applying the d^{μ}_0 operators on the ground state, which satisfy the Clifford algebra $\{d^{\mu}_0, d^{\nu}_0\} = \eta^{\mu\nu}$, i.e. the degeneration index $\alpha = 1,\ldots,32$ is in fact the index of a 32-component spinor in 10-dimensional Minkowski space-time. Just like in the Neveu-Schwarz sector, the entire Ramond sector is obtained by applying d^{μ}_n oscillators with $n<0$ to the ground state and contains only space-time fermions.

In the left-moving sector, the **bosonic ground state** $|0\rangle_{\mathrm{L}}$ does not have any particular properties. As expected, it transforms as a space-time scalar and all higher excitations gained by applying $\alpha_{\mathrm{L},n}$ for $n<0$ yield states that transform under respective tensor representations.

The excitations of the oscillators in turn influence the left- and right-moving "masses" of the states

$$\frac{1}{4}m_{\mathrm{L}}^2 = N_{\mathrm{L,bos}} + \frac{1}{2}p^I_{\mathrm{L}}p^I_{\mathrm{L}} - 1 \qquad \frac{1}{4}m_{\mathrm{R}}^2 = N_{\mathrm{R,bos}} + N_{\mathrm{R,ferm}} - a, \tag{7.6}$$

where a is a normal ordering constant, either being $a^{\mathrm{R}} = 0$ or $a^{\mathrm{NS}} = \frac{1}{2}$, and $N_{\mathrm{R,ferm}}$ must be chosen as either Ramond or Neveu-Schwarz, depending on the fermionic spin structure of the right-moving ground state. Only states satisfying the **level-matching condition** $m_{\mathrm{L}}^2 = m_{\mathrm{R}}^2$ are physical and have the respective **mass** $m^2 = m_{\mathrm{L}}^2 + m_{\mathrm{R}}^2$.

The states obtained cannot be supersymmetric as neither the number of bosonic and fermionic degrees of freedom is matching, nor is the spectrum tachyon-free. Define the **fermionic number operators**

$$F^{\mathrm{NS}}_{\mathrm{R}} := \sum_{r=\frac{1}{2}}^{\infty} b_{\mathrm{R},-r}\cdot b_{\mathrm{R},r} \qquad \text{and} \qquad F^{\mathrm{R}}_{\mathrm{R}} := \sum_{n=1}^{\infty} d_{\mathrm{R},-n}\cdot d_{\mathrm{R},n},$$

then the **GSO projection** can be simply stated as the projection onto the states with even fermion number. Thus, the **Fock space of physical R-moving states** takes the form

$$\mathcal{F}_{\mathrm{R,GSO}} := \mathcal{F}^{\mathrm{R}}_{\mathrm{R,GSO}} \oplus \mathcal{F}^{\mathrm{NS}}_{\mathrm{R,GSO}}$$

$$\text{where} \quad \begin{aligned} \mathcal{F}^{\mathrm{R}}_{\mathrm{R,GSO}} &:= \Big\{ |\psi;\mathrm{R}\rangle_{\mathrm{R}} \in \mathcal{F}^{\mathrm{R}} : (-1)^{F^{\mathrm{R}}_{\mathrm{R}}}|\psi;\mathrm{R}\rangle_{\mathrm{R}} = |\psi;\mathrm{R}\rangle_{\mathrm{R}} \Big\} \\ \mathcal{F}^{\mathrm{NS}}_{\mathrm{R,GSO}} &:= \Big\{ |\psi;\mathrm{NS}\rangle_{\mathrm{R}} \in \mathcal{F}^{\mathrm{NS}} : (-1)^{F^{\mathrm{NS}}_{\mathrm{R}}}|\psi;\mathrm{NS}\rangle_{\mathrm{R}} = |\psi;\mathrm{NS}\rangle_{\mathrm{R}} \Big\}. \end{aligned}$$

A Majorana-Weyl condition is imposed on the R-moving groundstate Dirac spinor $|0,\alpha;R\rangle_{\mathrm{R}}$, which reduces it from 32 to 8 components. This also reduces the groundstate to a Spin(8)-spinor in accordance to the behavior of massless particles. In particular, it yields the same

state	count	description
$\alpha^{\mu}_{\mathrm{L},-1}\lvert 0\rangle_{\mathrm{L}} \otimes \lvert q\rangle_{\mathrm{R}}$	1	10-dimensional $\mathcal{N}=1$ supergravity multiplet, where α^{μ}_{-1} refers to the eight transverse bosonic oscillators
$\alpha^{I}_{\mathrm{L},-1}\lvert 0\rangle_{\mathrm{L}} \otimes \lvert q\rangle_{\mathrm{R}}$	16	uncharged $\mathrm{E}_8 \times \mathrm{E}_8$ gauge bosons and corresponding gaugini in 16 $\mathcal{N}=1$ Yang-Mills supermultiplets
$\lvert p^{I}\rangle_{\mathrm{L}} \otimes \lvert q\rangle_{\mathrm{R}}$	480	charged $\mathrm{E}_8 \times \mathrm{E}_8$ gauge bosons and corresponding gaugini in 480 $\mathcal{N}=1$ Yang-Mills supermultiplets

TABLE 7.2. Massless supermultiplet content of the heterotic $\mathrm{E}_8 \times \mathrm{E}_8$-string.

number of bosonic and fermionic degrees of freedom in the ground states—a prerequisite for space-time supersymmetry. The GSO projection then ensures a space-time supersymmetric spectrum of states, i.e. it also discards states with odd fermion number. In the case of the heterotic string, 10-dimensional $\mathcal{N}=1$ supersymmetry can be established by further considerations.

7.8. Massless heterotic particle spectrum

Since excitations above the massless level result in particles of **Planck mass order** (10^{17} GeV)—which is not stressed in the equations above—the excited states of a string theory are usually neglected, since experimental detection of such states is far out of reach (the LHC is targeted to achieve collision energies up to 10^4 GeV). Thus, the massless heterotic spectrum can be derived from all solutions of the following equalities:

$$\begin{aligned}
0 = \tfrac{1}{4}m_{\mathrm{L}}^2 &= N_{\mathrm{L,bos}} + \tfrac{1}{2}p_{\mathrm{L}}^I p_{\mathrm{L}}^I - 1 &&\Longrightarrow && N_{\mathrm{L,bos}} = 1,\quad p_{\mathrm{L}}^I p_{\mathrm{L}}^I = 0 \quad \text{or}\\
& && && N_{\mathrm{L,bos}} = 0,\quad p_{\mathrm{L}}^I p_{\mathrm{L}}^I = 2\\
0 = \tfrac{1}{4}m_{\mathrm{R}}^2 &= N_{\mathrm{R,bos}} + N_{\mathrm{R,ferm}}^{\mathrm{R}} &&\Longrightarrow && N_{\mathrm{R,bos}} = 0,\quad N_{\mathrm{R,ferm}}^{\mathrm{R}} = 0\\
0 = \tfrac{1}{4}m_{\mathrm{R}}^2 &= N_{\mathrm{R,bos}} + N_{\mathrm{R,ferm}}^{\mathrm{NS}} - \tfrac{1}{2} &&\Longrightarrow && N_{\mathrm{R,bos}} = 0,\quad N_{\mathrm{R,ferm}}^{\mathrm{NS}} = \tfrac{1}{2}
\end{aligned}$$

In accordance with the general behavior of massless particles, this yields either a massless SO(8)-vector state or a massless Spin(8)-spinor state. For convenience, those massless states are usually denoted by a 4-component vector, where an underscore denotes all possible permutations of the respective components:

$$\begin{aligned}
\text{Neveu-Schwarz:}\quad q &= \left(\underline{\pm 1; 0, 0, 0}\right)\\
\text{Ramond:}\quad q &= \left(\pm\tfrac{1}{2}; \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\right) \text{ with even number of + signs}
\end{aligned} \tag{7.7}$$

The first component of the R-spinor actually gives the **4-dimensional chirality** of the state, with the signs corresponding to fig. 1.1 on page 2, i.e. left-handed particles have chirality $-\frac{1}{2}$, whereas $+\frac{1}{2}$ indicates right-handed particles. Using this vector q, both the R-moving Ramond and Neveu-Schwarz states can be written shorthand as $\lvert q\rangle_{\mathrm{R}}$.

In the left-moving bosonic sector the momenta p_{L}^I of the gauge degrees of freedom X_{L}^I yield a massless state provided $p_{\mathrm{L}}^I p_{\mathrm{L}}^I = 2$. Since the momenta p_{L}^I are elements of the (self-dual) root lattice $\Lambda_{\mathrm{E}_8\times\mathrm{E}_8} \cong \Lambda_{\mathrm{E}_8} \times \Lambda_{\mathrm{E}_8}$, and $v^2 = 0$ has exactly 240 solutions for $v \in \Lambda_{\mathrm{E}_8}^{\star} \cong \Lambda_{\mathrm{E}_8}$, the momenta p_{L}^I can be thought of as **internal quantum numbers** with 480 different states. The remaining massless states of the L-moving bosonic string come from ordinary oscillation excitations of the ground state, i.e. $\alpha^{I}_{\mathrm{L},-1}\lvert 0\rangle_{\mathrm{L}}$ for the excitations of the toroidal compactified components of the gauge degrees of freedom and $\alpha^{\mu}_{\mathrm{L},-1}\lvert 0\rangle_{\mathrm{L}}$ for the eight remaining transverse oscillators.

The actual massless physical states of the heterotic $\mathrm{E}_8 \times \mathrm{E}_8$-string arise from tensoring the left- and right-movers together. Depending on whether the right-moving superstring state

$|q\rangle_{\mathrm{R}}$ is a space-time vector or space-time spinor, the entire state is of the same type. This yields the massless particle content of tab. 7.2.

It is useful to investigate the fermionic description of the heterotic string, which provides another perspective on the emerging gauge groups. Following the work of Coleman (see [Col75]) and Mandelstam (see [Man75]), in a 2-dimensional framework a bosonic degree of freedom can equivalently be described by two Majorana(-Weyl) spinors. Since the string worldsheet is such a 2d space, one may describe the 16 bosonic degrees of freedom X_{L}^I for $I = 1, \ldots, 16$ via 32 fermions λ_{L}^A for $A = 1, \ldots, 32$. The original internal quantum numbers p^I (which arise as momenta of the compactified gauge degrees of freedom) are then described by applying the fermionic oscillators $\lambda^A_{\mathrm{L},-\frac{1}{2}}$ to some ground state. Of course, like for the other worldsheet fermions of a free superstring, one has to choose one of the two spin structures with either periodic (Ramond) or anti-periodic (Neveu-Schwarz) boundary conditions. In principle, this would yield 2^{32} possibilities for the gauge fermion's boundary conditions, but due to consistency conditions and permutations invariance of the independent fermions, only two different possibilities remain:

- All 32 fermions have the same boundary condition (either Ramond or Neveu-Schwarz). In this case, the gauge bosons and gaugini can be explicitly expressed as states

  $$\begin{array}{lll} \text{gauge bosons:} & \lambda^A_{\mathrm{L},-\frac{1}{2}}\lambda^B_{\mathrm{L},-\frac{1}{2}}|\mathrm{NS}\text{ or }\mathrm{R}\rangle_{\mathrm{L}} \otimes |\mathrm{NS}\rangle_{\mathrm{R}} & \\ & & \text{for } A, B = 1, \ldots, 32. \\ \text{gaugini:} & \lambda^A_{\mathrm{L},-\frac{1}{2}}\lambda^B_{\mathrm{L},-\frac{1}{2}}|\mathrm{NS}\text{ or }\mathrm{R}\rangle_{\mathrm{L}} \otimes |\mathrm{R}\rangle_{\mathrm{R}} & \end{array}$$

  Due to $\dim \Lambda^2\mathbb{R}^{32} = \binom{32}{2} = 496$ this gives rise to 496 states arising from the possible combinations of A and B. Those 496 states constitute the 496-dimensional adjoint representation of SO(32), such that SO(32) is the gauge group for this choice of spin structures.
- 16 of the 32 fermions obey the same boundary condition, i.e. 16 are Neveu-Schwarz and 16 are Ramond. In this context it is useful to introduce two separate sets of indices $R, S = 1, \ldots, 16$ and $P, Q = 1, \ldots, 16$, which allows for a simple splitting of the indices $A = 1, \ldots, 32$ after a possible rearrangement. Consider

  $$\begin{array}{lcl} \lambda^A_{\mathrm{L},-\frac{1}{2}}\lambda^B_{\mathrm{L},-\frac{1}{2}}|\mathrm{NS},\mathrm{NS}\rangle_{\mathrm{L}} \otimes |\mathrm{NS}\rangle_{\mathrm{R}} & \Longrightarrow & \binom{16}{2} = 120 \text{ states} \\ \lambda^P_{\mathrm{L},-\frac{1}{2}}\lambda^Q_{\mathrm{L},-\frac{1}{2}}|\mathrm{R},\mathrm{R}\rangle_{\mathrm{L}} \otimes |\mathrm{NS}\rangle_{\mathrm{R}} & \Longrightarrow & \binom{16}{2} = 120 \text{ states} \\ \lambda^A_{\mathrm{L},-\frac{1}{2}}\lambda^P_{\mathrm{L},-\frac{1}{2}}|\mathrm{NS},\mathrm{R}\rangle_{\mathrm{L}} \otimes |\mathrm{NS}\rangle_{\mathrm{R}} & \Longrightarrow & 16^2 = 256 \text{ states.} \end{array}$$

  There is an $\mathrm{SO}(16)\times\mathrm{SO}(16)$ symmetry, such that the first two sets of states constitute the adjoint representations $(120, 1)$ and $(1, 120)$. The last set of states shows the behavior of a 16d fermion, such that the 256 states transform as $(128, 0) \oplus (0, 128)$ of $\mathrm{Spin}(16) \times \mathrm{Spin}(16)$. It can be shown, that those particular representations combine to the adjoint representation of E_8, or rather that 248 of E_8 splits to $120 \oplus 128$ of Spin(16), where the first part corresponds to a SO(16)-representation. Thus, the 496 states constructed for the above choice of spin structure / boundary conditions give rise to the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge group of the heterotic string.

In the rest of the exposition the bosonic formulation will be used, as it is much more suited for what is about to follow.

7.9. Low-energy effective supergravity approximation

Restricted to its massless states, the heterotic $\mathrm{E}_8\times\mathrm{E}_8$-string is described by a 10d $\mathcal{N} = 1$ supergravity coupled to super-Yang-Mills theory with gauge group $\mathrm{E}_8 \times \mathrm{E}_8$. A general superspace formulation is found in [ADR86]. This description has two main virtues: the corresponding supergravity theory is directly formulated on space-time, i.e. the cumbersome world-sheet approach of string theory is not necessary, and it can be treated by the means of ordinary

field	type	description
G	$\Gamma(S^2\mathrm{T}^*\mathcal{M})$	space-time metric $(-+\cdots+)$ of $\mathcal{M}$
Φ	$C^\infty(\mathcal{M})$	real scalar dilaton field
$P \xrightarrow{\pi} M$		principal $\mathrm{E}_8 \times \mathrm{E}_8$-bundle over $\mathcal{M}$
A	$\Gamma(\mathrm{T}^*\mathcal{M} \otimes \operatorname{ad} P)$	gauge potential (connection) on P
F_A	$\Gamma(\Lambda^2\mathrm{T}^*\mathcal{M} \otimes \operatorname{ad} P)$	gauge field-strength (curvature) of A
χ	$\Gamma\big(S^+_{\mathbb{R}}(\mathcal{M}) \otimes \mathrm{T}^*\mathcal{M}\big)$	Majorana-Weyl gravitino
φ	$\Gamma\big(S^-_{\mathbb{R}}(\mathcal{M})\big)$	Majorana-Weyl dilatino
λ	$\Gamma\big(S^+_{\mathbb{R}}(\mathcal{M}) \otimes \operatorname{ad} P\big)$	Majorana-Weyl gaugino
H		axion 3-form, see text

TABLE 7.3. Fields in the heterotic low-energy effective supergravity action.

quantum field theory. However, while much information can be gained from this, the resulting low-energy supergravity theory is non-renormalizable due to the common problems encountered in canonical approaches to quantum gravity. Since the string-like extension of the fundamental object is no longer present, nothing regularizes the occurring divergences.

The actual derivation of the heterotic supergravity effective action is quite lengthy. In [BBS07, §8.1] a readable introduction to the general subject of supergravity approximations to various string theories with further reference is given. For the present case, the corresponding 10-dimensional **heterotic string supergravity effective action** is

$$\begin{aligned} &S_{\text{het,eff}}[G,\Phi,A,\chi,\varphi,\lambda,H] \\ &\quad= \frac{1}{2\kappa^2}\int_{\mathcal{M}} \mathrm{dvol}_G \Bigg[\mathrm{e}^{-2\Phi}\overbrace{\Big(R_G + 4(\nabla\Phi)^2 - \bar\chi_\alpha\gamma^{[\alpha}\gamma^{\beta}\gamma^{\gamma]}\nabla_\beta\chi_\gamma - \bar\varphi\not{D}\varphi\Big)}^{\text{10d } \mathcal{N}=1 \text{ supergravity term}} \\ &\qquad + \underbrace{\mathrm{e}^{-2\Phi}\left(\frac{\alpha'}{30}\operatorname{Tr}(F_A\wedge F_A) - \operatorname{Tr}(\bar\lambda\not{D}\lambda)\right)}_{\mathcal{N}=1 \text{ super-Yang-Mills term}} + \mathrm{e}^{-2\Phi}\left(-\frac{1}{3}\tilde H^2\right) + \left(\begin{smallmatrix}\text{further}\\ \text{terms}\end{smallmatrix}\right)\Bigg], \end{aligned} \tag{7.8}$$

with the fields listed in tab. 7.3. By construction the action does not include any massive (fermionic) states. However, such situations can be approximated by adding appropriate "further terms" as indicated in the above action. The term involving $\tilde H$ describes the Kalb-Ramond B-field coupled to the space-time metric and the gauge field. This so-called **axion 3-form** is described by the property

$$\mathrm{d}\tilde H = \alpha'\left(\operatorname{tr}(R_G\wedge R_G) - \frac{1}{30}\operatorname{Tr}(F_A\wedge F_A)\right), \tag{7.9}$$

where Tr refers to the "gauge trace" in the adjoint representation and tr to the "normal trace" in the fundamental representation (if applicable). Locally, $\tilde H$ can be described by

$$\tilde H = \mathrm{d}B + \alpha'\left(\mathrm{CS}(\nabla^G) - \frac{1}{30}\mathrm{CS}(A)\right), \tag{7.10}$$

involving the **Chern-Simons 3-form** of the metric Levi-Civita connection and the gauge connection, which can be interpreted as secondary characteristic classes involving ideas from topological quantum field theory.

Finally, it is necessary to describe the actual behavior of the fields in the effective supergravity action under local supersymmetry transformations. Given a Majorana-Weyl spinor ϵ, the corresponding symmetry transformation yields the following (bosonic) variations to the various fields:

$$
(7.11) \qquad \begin{cases} \text{metric:} & \delta_\epsilon G_{\mu\nu} = \bar{\epsilon}(\gamma_\mu \chi_\nu + \gamma_\nu \chi_\mu) + (\text{fermions})^2 \\ \text{dilaton:} & \delta_\epsilon \Phi = \bar{\epsilon}\varphi\Phi \\ \text{gauge field:} & \delta_\epsilon A_\mu = \bar{\epsilon}\gamma_\mu \lambda + (\text{fermions})^2 \\ \text{gravitino:} & \delta_\epsilon \chi_\mu = \nabla_\mu \epsilon - \frac{1}{4}\tilde{H}_{\mu\alpha\beta}\gamma^{[\alpha}\gamma^{\beta]}\epsilon + (\text{fermions})^2 \\ \text{dilatino:} & \delta_\epsilon \varphi = (\gamma^\mu \nabla_\mu \Phi)\epsilon + \frac{1}{24}\tilde{H}_{\mu\nu\rho}\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\epsilon + (\text{fermions})^2 \\ \text{gaugino:} & \delta_\epsilon \lambda = F_{\mu\nu}\gamma^{[\mu}\gamma^{\nu]}\epsilon + (\text{fermions})^2 \end{cases}
$$

This information is quite important for deriving ample conditions for partial supersymmetry breaking in the following chapters, however, the detailed structure of fermionic variations is unimportant. The fermionic terms and an outline of their derivation can be found in [CM83].

7.10. Anomaly cancellation

Quantum field theories containing chiral fermions are plagued by a very peculiar type of problem: anomalies, which represent the breakdown of a classical symmetry after the canonical quantization has been carried out, i.e. a classical symmetry might not be valid for the quantum theory, even after applying certain regularization techniques. Since only chiral fermion fields contribute, anomalies can only arise in even space-time dimensions $d = 2n$, where a Weyl-spinor splitting is possible. Furthermore, by a generalized version of the **Adler-Bardeen theorem** (cf. [AB69]), anomalies only arise in Feynman diagrams with $\frac{d}{2} + 1$ external legs attached to a single fermion loop. Thus, in 3+1 dimensions anomalies may be found in triangle diagrams, whereas in 9+1 dimensions hexagon diagrams can contribute anomalous terms, see fig. 7.4.

In general, anomalies can be classified in three different categories, which are related to the external boson legs attached to the fermion loop:

- **Gauge** or **Yang-Mills anomalies** arise if the external legs are Yang-Mills gauge vector fields and the underlying gauge symmetry is broken upon quantization.
- If the diffeomorphism or local Lorentz invariance does not survive the quantization process, the corresponding fermion-loop diagram with graviton legs gives an anomalous contribution. The absence of such **gravitational anomalies** is crucial for any theory of quantum gravity.
- General **global anomalies** can arise due to global problems of external currents.

Those anomalies can also appear in a mixed form, of course, and one speaks of **mixed anomalies** in this case.

The central property of anomalies is that—unlike the "common" infinites, which are treated by regularization and renormalization techniques—they cannot be controlled, i.e. they represent fundamental inconsistencies in a physical theory involving chiral fermions. A thorough introduction to the subject, which introduces anomalies as well as their relation to regularization and to conserved currents, is found in [AGW83] and [AG86].

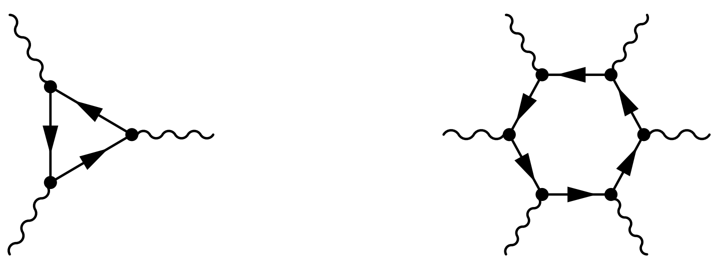

FIGURE 7.4. Possible anomalous Feynman diagrams in 4d and 10d space-time. The external legs are either gauge fields, gravitons or general external currents.

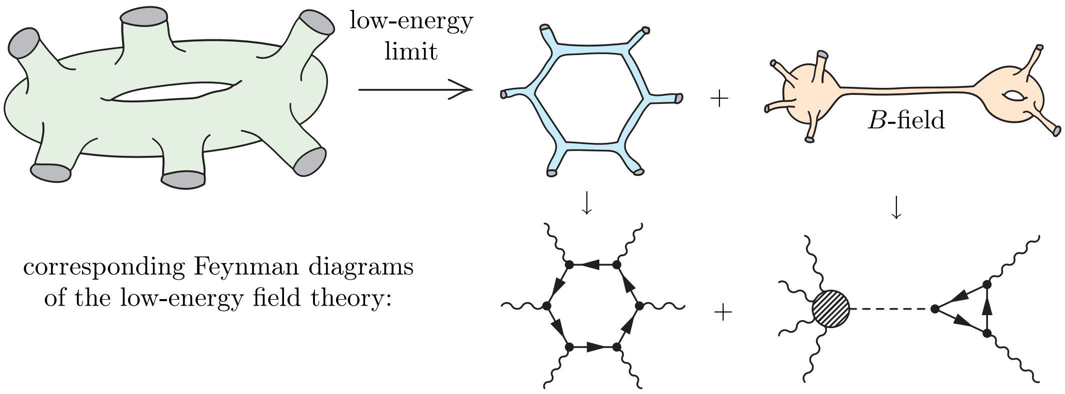

FIGURE 7.5. The exchange of the antisymmetric B-field along tree level diagrams in the Green-Schwarz mechanism yields a counter-term for the anomalous field theoretic hexagon diagram.

In 1984, inaugurating the so-called "first superstring revolution", Green and Schwarz found that in a rather miraculous way (see [GS84] and [GS85]) 10d supergravity coupled to super-Yang-Mills theory is anomaly-free for the gauge groups SO(32) or $\mathrm{E}_8 \times \mathrm{E}_8$. Let the gauge-field strength (curvature of the gauge connection A) be denoted as $F_A := \mathrm{d}A + \frac{1}{2}[A, A]$ and $R_G := \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$ be the curvature 2-form of the Levi-Civita connection associated to the space-time metric G.

There exists an formalism to compute anomalies in terms of polynomials of traces of R_G and F_A, see [BBS07, §5.4] or [Ura03] for a concise review. If ξ is a gauge parameter, then in general the anomalous variation of an effective 10d action can be written as

$$\delta_\xi S_{\text{eff}} = \int_{\mathcal{M}} G_{10},$$

where G_{10} is a 10-form on the 10d space-time $\mathcal{M}$. In general, G_{10} is a rather complicated term, but it can be efficiently represented as follows: Let I_{12} be a closed formal "12-form" on $\mathcal{M}$—of course, there are no 12-forms on a 10d space—constructed as a polynomial in R_G and F_A. Then locally $I_{12} = \mathrm{d}\omega_{11}$ holds for an formal "11-form" ω_{11}, and the gauge variation of ω_{11} is related to the anomalous 10-form G_{10} via

$$\delta_\xi \omega_{11} = \mathrm{d}G_{10}.$$

In the case of the low-energy 10d heterotic string approximation (7.8), there appears to be an anomalous term of the form

$$I_{12} \propto \left(\operatorname{tr} {R_G}^2 - \operatorname{tr} {F_A}^2\right) \operatorname{tr} {F_A}^4, \tag{7.12}$$

where the exponents are to be understood as ${R_G}^2 = R_G \wedge R_G$ with respect to the wedge product extension to Lie-algebra-valued differential forms. Due to the underlying string-structure of (7.8), there is a non-standard contribution achieved by stretching the canonical anomalous "hexagon tube diagram" as indicated in fig. 7.5, such that there appears an exchange of a massless boson—which turns out to be the antisymmetric Kalb-Ramond field—along a tree level diagram. The resulting anomaly cancellation of both terms is essentially due to the Chern-Simons coupling of the B-field to the space-time and gauge curvature, as expressed in (7.10).

For the gauge group SO(32) the trace in the adjoint representation is equal to 30 times the trace in the fundamental representation, i.e. $\operatorname{tr} {F_A}^2 = 30 \operatorname{Tr} {F_A}^2$, such that the modified Bianchy identity (7.9) reads

$$\mathrm{d}\tilde{H} = \alpha' \left(\operatorname{tr} {R_G}^2 - \operatorname{tr} {F_A}^2\right),$$

which closely resembles the first factor in the anomalous term (7.12). In the case of $E_8 \times E_8$ the adjoint representation—as the smallest non-trivial representation—equals the fundamental representation, and a similar line of reasoning gives the same term as the one above. A careful computation (which vastly exceedes the bounds of this work, see [GS84] and [GS85]) shows, that the observation is indeed correct and the anomalies cancel. This construction and the corresponding field theory interpretation, as depicted in fig. 7.5, is called the **Green-Schwarz mechanism**.

Besides the anomalies the Green-Schwarz mechanism deals with, one may be concerned about global anomalies. In [Wit85b] Witten proposed a rather beautiful geometrical interpretation of global gravitational anomalies by showing that the determinant line bundle $K_{\mathcal{M}} \xrightarrow{\pi} \mathcal{M}$ of the space-time carries a natural connection, whose curvature and holonomy encode the information of the anomalies. More precisely, the torsion of this connection represents the global gravitational anomalies, and in [Wit85a] this was applied to certain special cases. Freed extended those results in [Fre86] to arbitrary background space-times, which shows the complete anomaly freedom of the heterotic string.

The physical implications are particularly noteworthy: obviously, string theory crucially relies on the inclusion of gravity in order to avoid anomalies (i.e. fundamental inconsistencies)—without the metric G being a dynamic object of the theory, the outlined construction would not be possible. From a certain perspective, this can also be seen as one of the shortcomings of conventional field theory approaches to quantum gravity.

7.11. Finiteness of perturbative string theory

Complementary to the discussion of the non-renormalizability of SUGRA theories back in sec. 6.7, the ultraviolet properties of string theory are briefly investigated. One of the central problems regarding this issue is the complicated structure of the moduli space for an arbitrary worldsheet, which has to be integrated over in order to evaluate a scattering (or transition) amplitude. This becomes particularly involved for the spin structures and associated supermoduli in the supersymmetric string theories.

Nowadays, there are quite a number of different finiteness proofs available, which essentially solve the problem from different perspectives. The "classic" arguments are found in [Mar86, §3], [AMS88] and [Man92, §5]. The proof of Atick, Moore and Sen is rather technical and lengthy, but it does not require any deeper insight. In the original paper, a g-loop order diagram with n external legs is investigated for $g, n \in \mathbb{N}$, and the authors show that the corresponding amplitude is ambiguous due to a certain freedom of gauge choices. Since this ambiguity is in fact only a total derivative and can be expressed as a product of vacuum amplitudes and tadpole diagrams—which can be shown to vanish in this context—the finiteness of perturbative string theory is established to all orders. Since this result is available for almost 20 years, the finiteness issue of superstring theory is essentially settled.[j] A particularly noteworthy modern approach is found in [Ber04], where Berkovits uses his own pure spinor formalism to prove the finiteness of perturbative string theory in this context.

Nevertheless, all those results are obtained in a rather abstract or indirect fashion, whereas explicit and constructive proofs are rather limited. In fact, it took 17 years from the 1-loop proof—carried out by Green and Schwarz for the type-II superstring in [GS82] and by Gross et al. for heterotic strings in [GHMR86]—to arrive at the next order. In 2001 D'Hoker and Phong, who already investigated the subject in the 80s (see [DP86]), proved in a lengthy series of papers ([DP02a], [DP02b], [DP02c] and [DP02d]) that the type-II superstring and heterotic string are indeed finite up to 2-loop order.

[j] There are recurring arguments against the validity of those proofs. Since physical proofs are usually less rigorous compared to strict mathematical statements, there may indeed be loopholes in the arguments—however, despite many discussions regarding this issue, no such loophole has ever been found in two decades. Nevertheless, a proof directly in terms of supersymmetric σ-models with respect to arbitrary Riemannian surfaces would seem preferable to the author.

CHAPTER 8

Kaluza-Klein Mechanism and Smooth Compactification

The heterotic superstring can only be consistently formulated in 9+1 (and 25+1 for the bosonic part) space-time dimensions. However, this consistency condition strongly disagrees with the every-day observation of three spatial degrees of freedom. The chapter begins with a discussion of the possible dimensionality of space-time, which is of very general nature. Next, the Kaluza-Klein mechanism is developed in some detail (following [DEF$^+$99, II-Compact., lect. 1]), first by compactifying a single dimension on a circle, then in the more general case of a 6d internal manifold. This leads to an emergence of massless scalars and gauge fields. The results are used in the toroidal compactification of the heterotic string, where the effective 4d particle content is described in detail. Since this approach has very unphysical properties—no breaking of the supersymmetry—the remaining sections deal with the partial breaking of the effective 4d supersymmetry in order to arrive at the Calabi-Yau compactification, where global $\mathcal{N} = 1$ (or $\mathcal{N} = 2$) SUSY can be restored. An entire section is spent on the issue of "embedding the spin connection in the gauge degrees of freedom", as it will be of some importance in the upcoming orbifold chapters. The chapter closes with a short summary about the pros and cons of both developed compactification models, i.e. the toroidal and Calabi-Yau approach.

8.1. The case for 3+1 large space-time dimensions

Before the different types of smooth compactification of the heterotic string theory are discussed, it should be investigated what singles out a 3+1 dimensional space-time from the physical point of view. This question is a rather old one and was already considered by Kant in the 18th century, albeit from a purely philosophical point of view. The question is: How many time-like and spatial dimensions are viable for a universe at least remotely similar to the one observed? This question is often associated with the anthropic principle, but it can be refined in the prospect of modern physical understanding.

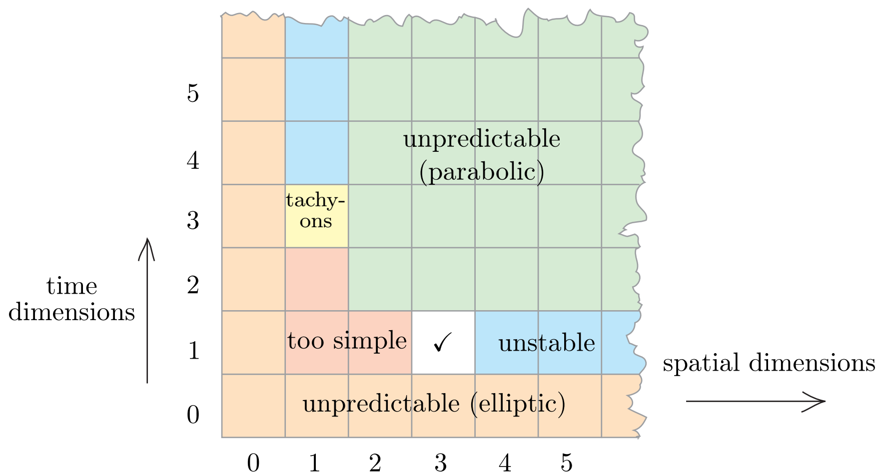

Figure 8.1. Stability properties and predictability in different space-time dimensions. Image reproduced from [Teg97].

Science agrees that the development of complex structures (e.g. nuclei, atoms, molecules, life, galaxies, etc.) strongly depends on stability, even if the relevant scales of stability can be very different. A quite simple but fundamental situation is the two-body problem, which is found both in the hydrogen atom (microscopic scale) and the solar system (macroscopic scale). In 1917 Ehrenfest considered this problem in arbitrary spatial dimensions and found that only for $d \leq 3$ stable solutions are found, as the "volume" of a d-dimensional sphere S^d depends on the radius r like $\frac{1}{r^{d-1}}$, i.e. the higher the dimension d becomes, the more sensitive the system will be to small radial disturbances. In 4 spatial dimensions, a small disturbance of a planets movement would be enough to let it either crash to its central star or to release it to outer space. The case $d = 1, 2$ on the other hand cannot be ruled out by such a strong result, however, it is usually argued, that one or two spatial dimensions do not offer enough degrees of freedom to develop complex structure.[a] This can be understood in terms of topological obstructions, as a two-dimensional space does not allow for over-crossings of lines, cf. [Teg97].

The number of time-like dimensions can be rephrased to a stability question along similar lines of reasoning. The elementary concepts of fundamental physics are energy and momentum conservation, which translate to particles guided by Lorentzian geodesics in the case of elementary particle physics. However, given an additional degree of time-like freedom (with the associated negative signature sign in contrast to the positive sign associated to spatial coordinates), the deformation of any curve in time-like directions actually shortens the length with respect to the metric. Thus, in space-times with more than one time-like dimension the most fundamental concepts of physics are no longer fulfilled. On the other hand, with no time dimension there is not any kind of evolution.

One can even refine the derived result of a necessarily 3+1 dimensional space-time, in order to allow for any kind of sophisticated structure: Without a symmetry breaking of the time inversion symmetry there is no notion of "before" and "after", such that the causality principle would not be present. In the prospect of P and CP violations in weak interactions, similar considerations might also be appropriate for the spatial dimensions.

8.2. Circular Kaluza-Klein mechanism

In 1926 the Kaluza-Klein mechanism was considered in order to unify classical electrodynamics and general relativity. The 5-dimensional pure gravity **Einstein-Hilbert action**

$$S[G] = \int_{\mathcal{M}^5} \mathrm{dvol}_G \, R_G \tag{8.1}$$

on a 4+1-dimensional space-time $\mathcal{M}^5$, where R_G is the Ricci scalar curvature of the metric G_{MN} and dvol_G the corresponding volume element, has a $\mathrm{Diff}(\mathcal{M}^5)$-symmetry. Varying the metric G yields the vacuum Einstein field equations, which reduce to the condition of Ricci flatness of the metric G_{MN} due to the absence of an energy-momentum source term. On a Riemannian product manifold $\mathcal{M}^5 = \mathbb{R}^4 \times S^1$ with coordinates (x, θ) a **5-dimensional 4d-flat product metric** can be defined by

$$G^{\mathrm{flat}}_{MN}(x, \theta) := \left(\begin{array}{c|c} \eta_{\mu\nu} & 0 \\ \hline 0 & r^2 \end{array}\right),$$

where r is the radius of the circular dimension. On the other hand, an arbitrary general 5d metric can be parameterized as

$$\begin{aligned} \mathrm{d}s^2 = G_{MN}\, \mathrm{d}z^M \, \mathrm{d}z^N &:= \mathrm{e}^{-\frac{\phi}{3}} \left(g_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu + \mathrm{e}^{\phi} (\kappa A_\mu \, \mathrm{d}x^\mu + \mathrm{d}\theta)^2 \right) \\ \Longrightarrow \qquad G_{MN}(x, \theta) &= \left(\begin{array}{c|c} \kappa^2 \mathrm{e}^{\frac{2\phi}{3}} A_\mu A_\nu + g_{\mu\nu} & \mathrm{e}^{\frac{2\phi}{3}} \kappa A_\mu \\ \hline \mathrm{e}^{\frac{2\phi}{3}} \kappa A_\nu & \mathrm{e}^{\frac{2\phi}{3}} \end{array}\right), \end{aligned}$$

[a] Abbott's famous novel *"Flatland"* vividly describes the problems and occurrences encountered in a lower-dimensional space-time. It is a remarkable fact, that it seems easy to imagine lower-dimensional spaces, whereas even a simple 4-dimensional sphere seems to be completely out of grasp.

wherein the 5d space-time metric G_{MN} was replaced by a 4d symmetric tensor $g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$, a 4d 1-form field $A_\mu\,dx^\mu$ and a scalar field ϕ. However, one should note that the 4-dimensional fields still depend on all 5-dimensional space-time coordinates (x,θ).

Since S^1 is a compact space and θ a periodic coordinate, the component fields can be expanded into discrete Fourier series

$$
\begin{aligned}
g_{\mu\nu}(x,\theta) &= \sum_{n=-\infty}^{\infty} g_{\mu\nu}^{[n]}(x)\mathrm{e}^{\mathrm{i}n\theta}\\
A_\mu(x,\theta) &= \sum_{n=-\infty}^{\infty} A_\mu^{[n]}(x)\mathrm{e}^{\mathrm{i}n\theta}\\
\phi(x,\theta) &= \sum_{n=-\infty}^{\infty} \phi^{[n]}\mathrm{e}^{\mathrm{i}n\theta},
\end{aligned}
\tag{8.2}
$$

where the superscript index $[n]$ refers to the n-th mode. The basic idea when investigating the low energy effective theory is to ignore all the excited modes, which implies that only the zero modes of the expansions are kept. This also removes the dependence on the compactified (periodic) coordinate θ. To consider excitations of the 5d metric, the **linearized metric fluctuations** are defined by

$$
h_{MN} := \tilde{h}_{MN} - \frac{1}{2}G^{RS}_{\text{flat}}\tilde{h}_{RS}G^{\text{flat}}_{MN} \qquad \text{with} \quad \tilde{h}_{MN} := G_{MN} - G^{\text{flat}}_{MN}.
$$

After fixing the Diff$(\mathbb{R}^4\times S^1)$ coordinate invariance by imposing the condition $\nabla^M h_{MN}=0$, which is called the **transverse gauge**, the original condition of Ricci-flatness of the metric can be rewritten as $R(G^{\text{flat}}+h)=0$. In turn, is equal to

$$
\Box^5_{G^{\text{flat}}} h_{MN}(x,\theta) = G^{\text{flat}}_{RS}\partial^R\partial^S h_{MN}(x,\theta) = 0
$$

when expanded to first order in h. After Fourier expanding $h_{MN}(x,\theta)$ with respect to the periodic coordinate θ, i.e.

$$
h_{MN}(x,\theta) = \sum_{n=-\infty}^{\infty} h^{[n]}_{MN}(x)\mathrm{e}^{\mathrm{i}n\theta}
$$

in the same fashion as (8.2), this yields the eigenfunction equations

$$
\left(\Box^4_\eta - \frac{n^2}{r^2}\right) h^{[n]}_{MN}(x) = 0,
$$

which can also be interpreted as 4-dimensional wave equations with mass $\frac{n}{r}$. It turns out, that below energies of order $\frac{1}{r}$, the effective theory behaves purely 4-dimensional. The low energy theory now only consists of the following fields, all depending just on the uncompactified coordinates of $\mathbb{R}^4$:

- a 4-dimensional metric $g_0 := g^{[0]}_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$
- a 1-form field $A_0 := A^{[0]}_\mu(x)\,\mathrm{d}x^\mu$
- a scalar field $\phi_0 := \phi^{[0]}(x)$.

The 4-dimensional **Kaluza-Klein effective action** of the massless (zero mode) fields then reads

$$
S_{\text{eff}}[g_0,A_0,\phi_0] = r\int_{\mathbb{R}^4}\mathrm{dvol}_{g_0}\left(\frac{1}{\kappa^2}R(g_0) - \frac{1}{4}\mathrm{e}^{\phi_0}(F_0)_{\mu\nu}(F_0)^{\mu\nu} - \frac{1}{6\kappa^2}(\nabla\phi_0)^2\right).
$$

The symmetries remaining from the original Diff$(\mathcal{M}^5)$-invariance of (8.1) in this effective action are those which commute with the S^1-action of the compactified dimension—other types of diffeomorphisms would mix the massless and massive (energy) modes, which were discarded in the low-energy effective theory. Since diffeomorphisms on $\mathcal{M}^5$ are generated by smooth vector fields $V \in \mathfrak{X}(\mathcal{M}^5) = \mathfrak{X}(\mathbb{R}^4 \times S^1)$, this can be made more explicit in the case at hand. Essentially, there are two types of compatible symmetries:

- Vector fields, which are independent of the compactified θ-coordinate, generate pure $\mathbb{R}^4$-diffeomorphisms. For such a $V \in \mathfrak{X}(\mathbb{R}^4)$, the corresponding field variations are[b]

metric:	$\delta_V g_{\mu\nu} = \mathcal{L}_V(g_{\mu\nu})$
1-form field:	$\delta_V A_\mu = \mathcal{L}_V(A_\mu)$
scalar field:	$\delta_V \phi = \mathcal{L}_V(\phi) = V\phi$

- Vector fields W, which originate from the coordinate transformations $x^\mu \mapsto x^\mu$, $\theta \mapsto \theta + \zeta^5(x)$. The corresponding effective field variations are

metric:	$\delta_W g_{\mu\nu} = 0$
1-form field:	$\delta_W A_\mu = -\frac{1}{\kappa}\partial_\mu \zeta^5(x)$
scalar field:	$\delta_W \phi = 0$

  Note that those transformation symmetry transformations turn the 1-form field A into an abelian U(1)-gauge field.

The outlined process—to derive a lower-dimensional effective field theory from a single field by discarding excited (massive) states—is called the **Kaluza-Klein mechanism**. As mentioned before, Kaluza and Klein originally used the arising U(1)-gauge field A_μ to unify general relativity with classical electrodynamics.

8.3. Six-dimensional Kaluza-Klein mechanism

In higher-dimensional situations, the basic concept remains the same, albeit the one-dimensional Fourier expansion is replaced by an expansion in harmonic functions. In the 10-dimensional pure gravity case—which corresponds to the action (8.1) for a 10d instead of a 5d space—the classical equations of motion for the metric G_{MN} again yield the condition of Ricci flatness. A product manifold $\mathcal{M}^{10} = \mathbb{R}^4 \times K^6$ with K^6 a compact smooth 6-dimensional manifold is assumed for the Kaluza-Klein mechanism, where the **10-dimensional 4d-flat product metric** with coordinates (x, y) is given by

$$G^{\text{flat}}_{MN}(x,y) := \left(\begin{array}{c|c} \eta_{\mu\nu} & 0 \\ \hline 0 & g_{mn}(y) \end{array}\right)$$

with a positive definite metric g_{mn} on K^6. To preserve Ricci-flatness, the compact space's metric g_{mn} has to be Ricci-flat—which has profound topological implications for K^6, as it was thoroughly discussed in chap. 5.

Again, using the linear fluctuations $h_{MN} := G_{MN} - G^{\text{flat}}_{MN}$ in transverse gauge, the Ricci-flatness condition $R_{MN}(G^{\text{flat}} + h) = 0$ to first order in h reduces to

$$\Box^{10}_{G^{\text{flat}}} h_{MN}(x,y) = \left(\Box^4_\eta + \Box^6_g\right) h_{MN}(x,y) = 0.$$

The expansion in Fourier modes is replaced by an expansion in harmonics of K^6, i.e. in functions $Y^{[k]}(y)$ satisfying $\Box^6_g Y^{[k]}(y) = -\lambda_k Y^{[k]}(y)$, where the d'Alambert operator equals the Laplacian due to the positive definite metric. This yields the expansions

$$h_{\mu\nu}(x,y) = \sum_k h^{[k]}_{\mu\nu}(x) Y_k(y) \qquad \text{satisfying } \left(\Box^4_\eta - \lambda_k\right) h^{[k]}_{\mu\nu}(x) = 0,$$

---

[b]Given a vector field $V \in \mathfrak{X}(M)$, the **Lie derivative** $\mathcal{L}_V$ is another type of differentiation, which is closely related to the exterior differentiation. Let x^μ be a local coordinate system for M and $T \in \Gamma(\mathrm{T}M^{\otimes r} \otimes \mathrm{T}^*M^{\otimes s})$ be a tensor field of type (r, s). The Lie derivative along V is then given in explicit terms as

$$\begin{aligned}\mathcal{L}_V(T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}) = V^\rho(\nabla_\rho T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}) - (\nabla_\rho V^{\mu_1})T^{\rho\mu_2\ldots\mu_r}{}_{\nu_1\ldots\nu_s} - \cdots - (\nabla_\rho V^{\mu_r})T^{\mu_1\ldots\mu_{r-1}\rho}{}_{\nu_1\ldots\nu_s} \\ + (\nabla_{\nu_1} V^\rho)T^{\mu_1\ldots\mu_r}{}_{\rho\nu_2\ldots\nu_s} + \cdots + (\nabla_{\nu_s} V^\rho)T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_{s-1}\rho}\end{aligned}$$

In particular, for a smooth function $f \in C^\infty(M)$ the Lie derivative is simply equal to the directional derivative, i.e. $\mathcal{L}_V(f) = Vf$.

10d fields			4d fields
10d gauge vector field (8b)	$\Longrightarrow$	1	4d gauge vector field (2b)
		6	4d scalars (6b)
10d gaugino (8f)	$\Longrightarrow$	4	4d gaugini (8f)
10d $\mathcal{N}=1$ SYM	$\Longrightarrow$	1	4d $\mathcal{N}=4$ SYM

TABLE 8.1. Particle reduction of 10d $\mathcal{N}=1$ SYM multiplets upon Kaluza-Klein compactification on T^6.

i.e. the modes $h^{[k]}_{\mu\nu}(x)$ are 4-dimensional waves of mass $\sqrt{\lambda_k}$. Similarly, the modes in the expansion

$$h_{mn}(x,y) = \sum_k \phi^{[k]}(x) Y^{[k]}_{mn}(y) \qquad \text{satisfying } \left(\Box^4_\eta - \lambda'_k\right)\phi^{[k]}(x) = 0$$

are also 4d waves of mass $\sqrt{\lambda'_k}$, where λ'_k is the negative eigenvalue of the corresponding K_6-harmonic, i.e. $\Box^6_g Y^{[k]}_{mn}(y) = -\lambda'_k Y^{[k]}_{mn}(y)$. As before, only the massless modes are kept, i.e. all modes that correspond to K_6-harmonics satisfying $\Box^6_g Y^{[k]}(y) = 0$, which gives symmetric tensors $h^{[k]}_{\mu\nu}(x)$ and a number of scalar fields $\phi^{[k]}(x)$. These scalars can be understood as parameterizing the space of Ricci-flat metrics, i.e. they describe the geometry of the internal space.

The same principles can also be applied to fermions. After choosing a spin-structure on $\mathbb{R}^4 \times K^6$, the 10d Dirac operator

$$\not{D}^{10}_G : \Gamma\big(S^{\pm}(\mathbb{R}^4 \times K^6)\big) \longrightarrow \Gamma\big(S^{\mp}(\mathbb{R}^4 \times K^6)\big)$$

with respect to the product metric $G = \eta \times g$ can be split to $\not{D}^{10}_G = \not{D}^4_\eta + \not{D}^6_g$, such that the corresponding Dirac equation reads $\mathrm{i}(\not{D}^4_\eta + \not{D}^6_g)\psi = 0$. The solutions are of the general product form

$$\psi(x,y) = \sum_k \psi^{[k]}_4(x)\psi^{[k]}_6(y),$$

where $\mathrm{i}\not{D}^4\psi^{[k]}_4(x) = \lambda'_k\psi^{[k]}_4(x)$ and $\mathrm{i}\not{D}^6\psi^{[k]}_6(y) = -\lambda'_k\psi^{[k]}_6(y)$ are the respective eigenvector (or rather eigenspinor) equations, which in turn are just Dirac equations for particles with mass $\pm\lambda'_k$. As in the case of bosonic fields, the massive fluctuations are discarded and only the zero mode fermions on the internal space K^6 are kept.

Further information on Kaluza-Klein compactifications in the context of supergravity theories in various dimension can be found in [DNP86].

8.4. Toroidal compactification of heterotic strings

In the construction of the heterotic string (see chap. 7) the 16 unmatched bosonic fields in the L-moving part were toroidal compactified using an even, self-dual lattice of signature $(16,0)$. The same principle can also be extended to the R-moving superstring part, such that the 10-dimensional heterotic string is reduced to an 4-dimensional effective theory after compactification on T^6, see fig. 8.2. In order to be consistently defined on the 6-torus, the closed-string boundary conditions are modified to

$$X^i(\tau,\sigma+2\pi) = X^i(\tau,\sigma) + 2\pi W^i \qquad \text{for } i = 1,\ldots,6$$

where W represents a winding of the string around the internal torus. The boundary conditions of the fermionic fields remain unaffected.

The starting point is the heterotic string restricted to its massless states, i.e. the heterotic supergravity approximation (7.8) developed in the last chapter, which consists of a 10d graviton $G_{\mu\nu}$, a scalar dilaton Φ, a 2-form $B_{\mu\nu}$, gauge potentials A_μ, a Majorana-Weyl gravitino χ_μ, dilatino φ and gaugini λ. Those fields are contained in a single 10d $\mathcal{N}=1$ SUGRA and

10d fields			**4d fields**
10d graviton (35b)	$\Longrightarrow$	1	4d graviton (2b)
		6	4d graviphotons (U(1)-gauge vector fields) (12b)
		21	4d scalars (21b)
10d B-field (28b)	$\Longrightarrow$	1	4d B-field (pseudoscalar) (1b)
		6	4d U(1)-gauge vector fields (12b)
		15	4d scalars (15b)
10d dilaton (1b)	$\Longrightarrow$	1	4d dilaton (1b)
10d gravitino (56f)	$\Longrightarrow$	4	4d gravitini (8f)
		24	4d gaugini (48f)
10d dilatino (8f)	$\Longrightarrow$	4	4d dilatini (8f)
10d $\mathcal{N}=1$ SUGRA	$\Longrightarrow$	1	4d $\mathcal{N}=4$ SUGRA (incl. 6 U(1)-graviphotons)
		6	4d $\mathcal{N}=4$ U(1)-SYM

TABLE 8.2. Particle reduction of the 10d $\mathcal{N}=1$ SUGRA multiplet upon Kaluza-Klein compactification on T^6. The complete particle contents of the multiplets are listed in chap. 6. Note, that the antisymmetric tensor $B_{\mu\nu}$ in 4d is equivalent to a pseudoscalar, which gives the second dilaton found in the 4d $\mathcal{N}=4$ SUGRA multiplet.

sixteen 10d $\mathcal{N}=1$ $\mathrm{E}_8\times\mathrm{E}_8$-SYM multiplets, which are listed in chap. 6. The further 480 charged $\mathrm{E}_8\times\mathrm{E}_8$ gauge bosons are left out of this consideration, as they arise in a different manner from internal quantum numbers.

Upon compactification on a 6-torus, the Kaluza-Klein mechanism has the effect described in tab. 8.1 and tab. 8.2 on the fields of the SUGRA-multiplet and SYM-multiplets. Of course, the geometry of the T^6-torus is not arbitrary. In principle, one can also consider different tori for the left- and right-moving part of the heterotic string. Together with the T^{16}-torus of the L-moving gauge components, the most general approach is based on the choice of an asymmetric torus lattice $\Lambda_{22,6}$, called a **Narain lattice**. The same consistency considerations as before (recall sec. 7.5) require this lattice to be even and self-dual. The general approach also affects the gauge symmetry of the compactified theory, such that in general only a $\mathrm{U}(1)^{22}\times\mathrm{U}(1)^6$ gauge symmetry will remain in the 4d theory—however, there may be particular lattices that give larger gauge groups. The reduction process can be summarized as follows:

	heterotic 10d supergravity			T^6-comp. 4d theory
1	10d $\mathcal{N}=1$ SUGRA multiplet	$\Longrightarrow$	1	4d $\mathcal{N}=4$ SUGRA multiplet
16	10d $\mathcal{N}=1$ SYM multiplets		22	4d $\mathcal{N}=4$ SYM multiplets,

where the additional $\mathrm{U}(1)^6$-gauge symmetry is for the graviphotons inside the 4d $\mathcal{N}=4$ SUGRA multiplet, see tab. 6.3 on p. 66. The actual—quite lengthy—derivation of those results is detailed in [Ort04, §16.5].

Obviously, the toroidal compactification of either the SO(32)- or $\mathrm{E}_8\times\mathrm{E}_8$-heterotic string yields an $\mathcal{N}=4$ supersymmetric 4-dimensional theory. This can be explained by the following fact, circumventing the detailed derivation: due to the compactification, the fermionic part of the SUGRA multiplet can be split like

$$\alpha^{\mu}_{\mathrm{L},-1}|0\rangle_{\mathrm{L}}\otimes\Big|\overbrace{\pm\tfrac{1}{2}}^{\text{4d chirality}};\underbrace{\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}}_{\text{internal quantum numbers}}\Big\rangle_{\mathrm{R}}\ \text{with even number of + signs,}$$

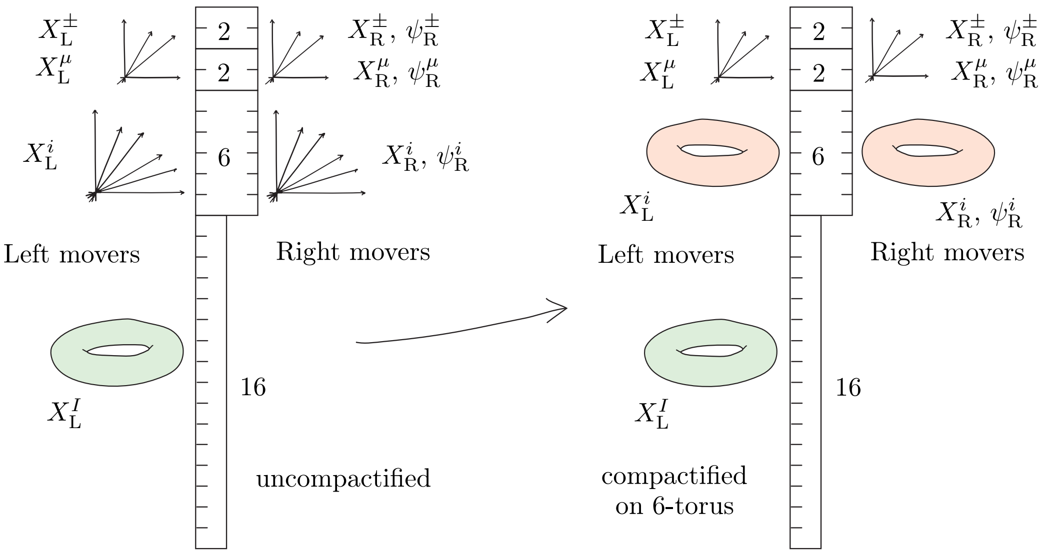

FIGURE 8.2. Toroidal compactification of the heterotic string.

i.e. the three “spin state” numbers associated with the toroidally compactified dimensions are treated as internal quantum numbers. Since only the combinations

$$
\begin{array}{c|ccc}
+ & + & + & + \\
+ & + & - & - \\
+ & - & + & - \\
+ & - & - & +
\end{array}
\qquad \text{and} \qquad
\begin{array}{c|ccc}
- & - & - & - \\
- & + & + & - \\
- & + & - & + \\
- & - & + & +
\end{array}
$$

satisfy the requirement of an even number of $+$ signs, for either choice of 4d chirality—which is fixed by the Majorana-Weyl condition—there are four internal quantum states (valid combinations of the three internal quantum numbers). In 4 dimensions the Weyl condition for the gravitini has to be dropped, such that a 4d gravitino consists of both left- and right-handed parts, the compactified theory has four gravitini, i.e. $\mathcal{N} = 4$ supersymmetry. The significance of this information lies in the fact, that supersymmetric theories are non-chiral if $\mathcal{N} > 1$, cf. chap. 6.

In the toroidal compactification process a large number of massless scalars was generated in the Kaluza-Klein mechanism:

graviton reduction:	21	
B-field reduction:	15	
reduction of 16 gauge fields:	96	(6 each)
	132	massless scalars

Each of those massless scalars can take an arbitrary vacuum expectation value and is called a **moduli field** or **scalar**. Thus, the toroidal compactification of the heterotic SO(32)- or $\mathrm{E}_8 \times \mathrm{E}_8$-string down to a 3+1-dimensional is described by 132 continuous parameters, and the parameter space of all the 132 moduli fields is called the **moduli space**. It can be described as the quotient space (see [Kir07, §9.1])

$$\mathcal{M}_{\mathrm{het}}^{4\mathrm{d},T^6} = \frac{\mathrm{O}(22,6;\mathbb{R})}{\mathrm{O}(22;\mathbb{R}) \times \mathrm{O}(6;\mathbb{R})} \Big/ \mathrm{O}(22,6;\mathbb{Z}),$$

which locally is a 132-dimensional manifold with singularities (i.e. an orbifold).[c] From a converse perspective, the moduli fields describe the geometry of the compactified dimensions, i.e. the shape of the 6-torus and the gauge degrees of freedom. Thus, $\mathcal{M}_{\text{het}}^{4\text{d},T^6}$ actually parameterizes the space of Narain lattices. As mentioned, for a generic point of this moduli space, only a $\mathrm{U}(1)^{22} \times \mathrm{U}(1)^6$ gauge symmetry remains in the corresponding effective theory—only for certain "critical" points larger gauge groups are obtained. Obviously, there is a great degree of arbitrariness involved in this construction.

Note the drastic change of the moduli-space upon toroidal compactification: The uncompactified 10d heterotic string only has a (discrete) choice of the gauge group, i.e.

$$\mathcal{M}_{\text{het}}^{10\text{d}} = \big\{\mathrm{SO}(32), \mathrm{E}_8 \times \mathrm{E}_8\big\},$$

whereas the distinction between those two theories vanishes in lower dimensions (upon toroidal compactification) and one is left with a connected 132-dimensional continuous moduli-space $\mathcal{M}_{\text{het}}^{4\text{d},T^6}$ in the case of T^6-compactification. Those moduli fields have an actual physical effect in the form of an (undesired) additional force. Chap. 11 spends a few words on this issue.

8.5. Conditions for 4d minimal supersymmetry

In order to arrive at an even remotely realistic 4-dimensional theory, the non-chiral $\mathcal{N} = 4$ supersymmetry has to be broken down to chiral $\mathcal{N} = 1$ supersymmetry—and in principle a process has to be developed, such that at low energies no trace of supersymmetry becomes visible, but this issue will be neglected in the following.

The starting point is again a 10-dimensional product space-time $\mathbb{R}^4 \times K^6$ with a Ricci-flat compact 6-dimensional space K^6, which was already considered in the context of the general Kaluza-Klein mechanism in sec. 8.3. As it was stressed there, the remaining symmetries of the compactified effective theory are those symmetries of the original action which do not mix massless and the discarded massive states. The central difference is, that the relevant spinorial supersymmetry generators are of antisymmetric (odd, fermionic) nature, in contrast to the (even, bosonic) vector fields that induce space-time diffeomorphisms. Nevertheless, the condition on a generator ϵ to induce an unbroken symmetry (in either case) is, that the variation $\delta_\epsilon \phi_\alpha$ is trivial for all fields ϕ_α of a given theory.

Since supersymmetry in 4 dimensions is generated by Weyl spinors, the first important (purely mathematical) observation is the splitting

$$S^+(\mathbb{R}^4 \times K^6) = \big[S^+(\mathbb{R}^4) \otimes S^+(K^6)\big] \oplus \big[S^-(\mathbb{R}^4) \otimes S^-(K^6)\big] \tag{8.3}$$

of the positive-chirality Weyl spinor bundle over the considered product space-time. Thus, any 10d Majorana-Weyl spinor ϵ on $\mathbb{R}^4 \times K^6$ is of the general form

$$\epsilon(x,y) = \sum_i \tilde{\epsilon}_+^i(x) \otimes \eta_+^i(y) + \text{c. c.}, \qquad \text{where} \qquad \begin{array}{l} \tilde{\epsilon}_+^i \in \Gamma\big(S^+(\mathbb{R}^4)\big) \\ \eta_+^i \in \Gamma\big(S^+(K^6)\big) \end{array} \tag{8.4}$$

are the separate 4d and 6d spinors on the external and internal space, and "c. c." stands for "complex conjugate". It is important to remember, that (nonzero) Weyl spinors of either chirality on $\mathbb{R}^4$ are not Poincaré-invariant—only the adjointment of both chiralities in the

[c]In general, the topology and geometry of a moduli space (of any type of compactification) is of a rather complicated nature, as it can contain singularities of various nature. Often the possible values of the moduli fields constitute a manifold or orbifold, but in particularly tough cases may degenerate to varieties (e.g. crossings of lines). Thus, in the physics literature, one often finds the simplified description that a portion of the moduli space is (locally) described by n parameters, which altogether avoids to mention any of the problematic situations. It is rather fortunate that in the case of 6d toroidal compactifications an explicit description of the moduli space is available.

form of a Dirac spinor is invariant.[d] Due to the splitting (8.3) there are no non-zero Poincaré-invariant Weyl spinors on the 10d space-time.

Thus, in order to satisfy the primary requirement of any 4-dimensional quantum field theory—Poincaré symmetry—all the fermionic fields (gravitino χ_μ, dilatino φ, gaugini λ) must be trivial. This immediately provides the SUSY-invariance of all bosonic fields, as

$$\delta_\epsilon(\text{bosonic field}) = (\text{bosonic term}),$$

and any bosonic product containing the fermionic symmetry parameter ε must also involve another fermionic field, c.f. (7.11). Thus, only the variations of the fermionic superpartners have to be investigated, i.e. the **equations for unbroken supersymmetry** reduce to

$$\begin{aligned} \text{gravitino:} \quad & \delta_\epsilon \chi_\mu = \nabla_\mu \epsilon - \tfrac{1}{4} \tilde{H}_{\mu\nu\rho} \gamma^{[\nu} \gamma^{\rho]} \epsilon = 0 \\ \text{dilatino:} \quad & \delta_\epsilon \varphi = \left(\gamma^\mu \nabla_\mu \Phi\right) \epsilon + \tfrac{1}{24} \tilde{H}_{\mu\nu\rho} \gamma^{[\mu} \gamma^\nu \gamma^{\rho]} \epsilon = 0 \\ \text{gaugini:} \quad & \delta_\epsilon \lambda = F_{\mu\nu} \gamma^{[\mu} \gamma^{\nu]} \epsilon = 0, \end{aligned} \tag{8.5}$$

where the $(\text{fermions})^2$-terms found in (7.11) were dropped due to their triviality as implied by Poincaré invariance.

In particular, the conditions imply that the gauge field A_μ, the axion 3-form $\tilde{H}_{\mu\nu\rho}$ and the dilaton Φ must be determined, such that there are exactly four SUSY-generating 4d Majorana spinor components in order to achieve 4d $\mathcal{N} = 1$ supersymmetry in the effective theory.[e]

8.6. No-flux solution to the simple supersymmetry conditions

To simplify the conditions (8.5) for unbroken 4d $\mathcal{N} = 1$ supersymmetry, a central step is to assume the axion 3-form $\tilde{H}$ to vanish, which will be dealt with in sec. 8.7. Due to the local structure of $\tilde{H}$, cf. (7.10), this implies that the antisymmetric Kalb-Ramond field B is described by a closed 2-form. Furthermore, the dilaton is assumed to be a constant function $\Phi = \Phi_0$. This leaves the two equations

$$\begin{aligned} \text{gravitino:} \quad & \delta_\epsilon \chi_\mu = \nabla_\mu \epsilon = 0 \\ \text{gaugini:} \quad & \delta_\epsilon \lambda = F_{\mu\nu} \gamma^{[\mu} \gamma^{\nu]} \epsilon = 0, \end{aligned} \tag{8.6}$$

both of which have profound implications for the geometry of the compact space K^6. If the particular 10d Majorana-Weyl spinor (8.4) is inserted into these equations, the gravitino condition yields two sets of equations:

$$\text{external coords. } (\mu = 0, \ldots, 3): \qquad \delta_\epsilon \chi_\mu = \sum_i \underbrace{\left(\partial_\mu \tilde{\epsilon}^i_+\right)}_{\Longrightarrow\ \tilde{\epsilon}_+ \text{ must be constant}} \otimes \eta^i_+ + \text{c. c.} = 0$$

$$\text{internal coords. } (m = 4, \ldots, 9): \qquad \delta_\epsilon \chi_m = \sum_i \tilde{\epsilon}^i_+ \otimes \underbrace{\left(\nabla_m \eta^i_+\right)}_{\Longrightarrow\ \eta_+ \text{ must be parallel}} + \text{c. c.} = 0$$

Since the 4d Weyl spinor $\tilde{\epsilon}_+$ has to be constant, the effective 4d theory can only be globally supersymmetric, i.e. there will not be a 4d $\mathcal{N} = 1$ supergravity. The $\mathcal{N} = 1$ supersymmetry condition (under the assumptions $\tilde{H} = 0$ and $\Phi = \Phi_0$) is then simplified to the search for internal parallel spinors, i.e. non-zero Weyl spinors $\eta^i_+ \in \Gamma\big(S^+(K^6)\big)$ satisfying

$$\nabla_m \eta^i_+(y) = 0.$$

[d]More precisely, Weyl spinors in even dimension are only invariant under the proper orthochronous Lorentz group $\mathcal{L}^\uparrow_+$, but not under the improper transformations (e.g. parity reflection) found in the full Lorentz group $\mathcal{L}$. Since the Poincáre group $\mathcal{P} = \mathcal{L} \ltimes \mathbb{R}^4$ includes the full Lorentz group, Weyl spinors are not invariant under the Poincaré group.

[e]Note that even after solving the supersymmmetry conditions (8.5) it remains unclear, whether there might be additional conditions for the equations of motion to be satisfied. This issue will not be discussed here—suffice to say, that they are indeed satisfied for the case in question.

The second equation in (8.6)—ensuring local SUSY invariance for the gaugino λ—is automatically satisfied by the embedding of the spin connection (see next section) into the gauge degrees of freedom and the conditions already imposed on ϵ: Using the usual identification $\Lambda^\bullet(\mathbb{R}^4 \times K^6) \cong \mathrm{C}\ell(\mathbb{R}^4 \times K^6)$ of the exterior algebra with the canonical Clifford bundle, the action of a differential form on the spinor ϵ is defined. The vanishing of the action of the gauge field-strength (curvature) 2-form F on ϵ is then equivalent to the vanishing of the action of the Riemannian curvature 2-form R on ϵ, i.e.

$$\underbrace{F_{\mu\nu}\gamma^{[\mu}\gamma^{\nu]}\epsilon = 0}_{\substack{\text{condition on gauge}\\ \text{connection}}} \quad \overset{\text{“embedding”}}{\Longleftrightarrow} \quad \underbrace{R_{\mu\nu\rho}{}^{\sigma}\gamma^{[\mu}\gamma^{\nu]}\epsilon = 0}_{\substack{\text{condition on spin}\\ \text{connection}}}.$$

The latter equation, however, is already satisfied due to ϵ being a parallel spinor, as required by the gravitino condition in (8.6).

As it was already discussed at length in chap. 5, the existence of a parallel spinor has profound implications for the topology and geometry of the internal space K^6. In particular, after normalizing a supposed parallel spinor η_+ to unit length, a complex structure

$$J_m{}^n := \mathrm{i}(\eta_+)^\dagger \gamma_{[m}\gamma^{n]}\eta_+$$

can be defined on the internal space K^6, i.e. the internal space must be a Ricci-flat Kähler manifold in order to yield an effective 4d $\mathcal{N} = 1$ supersymmetric theory after compactification. Due to the existence of the parallel spinor η_+, it can be shown (see [DEF+99, p. 1110] or [McI88, §4] for a much more detailed account on this delicate issue) that the internal space has SU(3)-holonomy.[f]

Thus, simple supersymmetry in the effective theory requires a Ricci-flat Kähler-manifold with SU(3)-holonomy. By definition, such a manifold in 6 real / 3 complex dimensions is called a **Calabi-Yau threefold**. The mathematical properties of such spaces where discussed at length in chap. 5.

8.7. Embedding of the spin connection in the gauge degrees of freedom

As shown, the embedding of the spin connection in the gauge connection is vital to the construction outlined in the previous section. The global vanishing of the axion 3-form field $\tilde{H}$ was a critical simplification step in the derivation of the 4d $\mathcal{N} = 1$ SUSY conditions. Since the description (7.10) of $\tilde{H}$ only holds locally, the globally defined modified Bianchi identity (7.9) is considered again in the form

$$\mathrm{d}\tilde{H} = \alpha' \left(\operatorname{tr} R_G{}^2 - \operatorname{tr} F_A{}^2\right).$$

Note that $\mathrm{d}\tilde{H} = 0$ allows for a global vanishing of $\tilde{H}$. Since $\mathrm{d}^2\tilde{H} = 0$ implies $\mathrm{d}(\operatorname{tr} R_G{}^2) = \mathrm{d}(\operatorname{tr} F_A{}^2)$, i.e. both 4-forms $\operatorname{tr} R_G{}^2$ and $\operatorname{tr} F_A{}^2$ have the same “boundaries” in the cohomological sense, the vanishing of $\mathrm{d}\tilde{H}$ is equivalent to

$$\left[\operatorname{tr} R_G{}^2\right] = \left[\operatorname{tr} F_A{}^2\right] \in H^2(\mathcal{M}^{10})$$

in terms of cohomology groups. Using results of Chern-Weil theory (i.e. the description of characteristic classes in terms of polynomials of the curvature), this can be restated as

$$c_2(\mathrm{T}\mathcal{M}) = c_2(P), \tag{8.7}$$

where $\mathrm{T}\mathcal{M} \xrightarrow{\pi} \mathcal{M}$ is the tangent bundle of the 10d space-time $\mathcal{M}^{10} = \mathbb{R}^4 \times K^6$ and $P \xrightarrow{\pi} \mathcal{M}$ the principal $\mathrm{E}_8 \times \mathrm{E}_8$-bundle.

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[f] Note that this result on the holonomy is neither implied by Berger's classification nor Wang's theorem from chap. 5—one could speculate from Berger that a SU(3)-holonomy manifold is the only Ricci-flat Kähler in 6 dimensions or by assuming from Wang that a SU(3)-holonomy manifold is the only 6d manifold admitting parallel spinors—as the important prerequisite of a simply connected space K^6 is not satisfied in general. Those two theorems essentially only provide information for the restricted holonomy group, as it was already stressed in chap. 5. See [McI88] for a more involved discussion.

Thus, the requirement of a vanishing axion 3-form has profound implications for the topology and geometry of the gauge bundle in order to give equal second Chern classes for the tangent and gauge bundle. The conditions on the gauge curvature (for a given metric of space-time) implied by (8.7) are very difficult to solve. Using the **Donaldson-Uhlenbeck-Yau theorem**, this leads to existence equations of stable holomorphic vector bundles—a very difficult issue in mathematics.

Fortunately, there exists a simple solution by taking a canonical sub-bundle of $P \xrightarrow{\pi} \mathcal{M}$ as follows: Treat $K \subset \mathcal{M}$ as a submanifold, then define the principal $\mathrm{E}_8 \times \mathrm{E}_8$-bundle

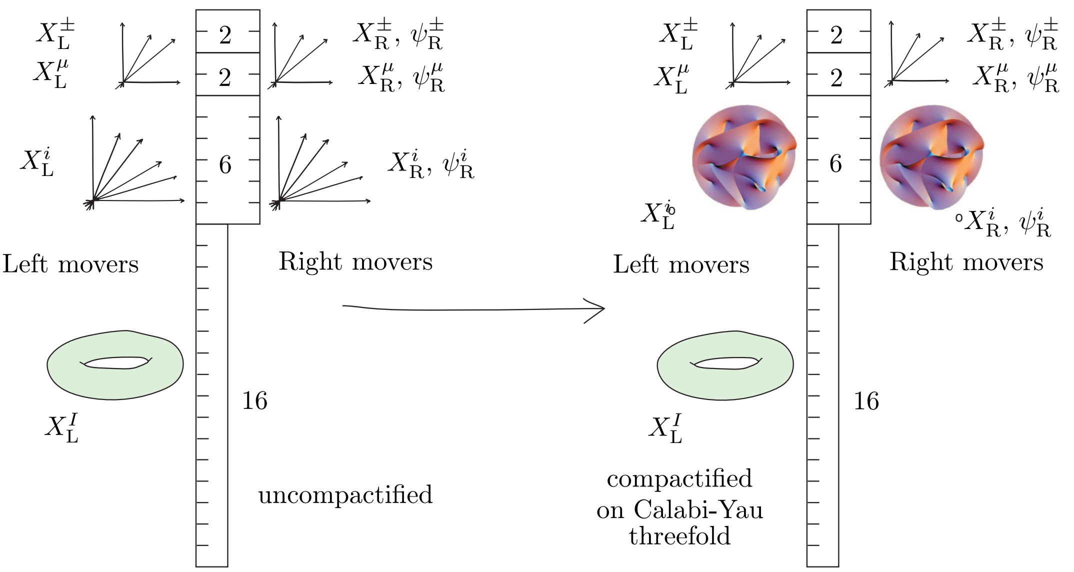
$$P_K := P|_K : \pi^{-1}(K) \xrightarrow{\pi} K.$$

Let $\mathrm{Hol}(K) \xrightarrow{p} K$ be the holonomy frame bundle of K, i.e. a principal SU(3)-bundle. In the absence of global topological obstructions (see [McI88, §6] for details), the holonomy bundle $\mathrm{Hol}(K) \xrightarrow{p} K$ can be embedded as a sub-bundle into the gauge bundle $P_K \xrightarrow{\pi} K$.

Now the "spin connection"—which in this description is in fact a connection on $\mathrm{Hol}(K)$—and the gauge connection can be related as follows: Let A be a connection (gauge field) on the gauge bundle $P \xrightarrow{\pi} \mathcal{M}$, such that the restriction $A|_K$ gives a connection on $P|_K \xrightarrow{\pi} K$. On the other hand, let ∇_K be the Levi-Civita connection on $\mathrm{T}K \xrightarrow{\tau} K$, which defines a corresponding connection $\tilde{\nabla}_K$ on the holonomy bundle $\mathrm{Hol}(K)$. Since $\mathrm{Hol}(K)$ is a sub-bundle of the gauge bundle P_K, the "embedding" of the spin connection in the gauge connection is expressed by

$$\overbrace{A|_K}^{\text{gauge connection}} = \underbrace{\tilde{\nabla}_K}_{\substack{\text{holonomy connection} \\ \text{("spin connection")}}} .$$

In order to preserve the embedding of the holonomy bundle, the original gauge symmetry $\mathrm{E}_8 \times \mathrm{E}_8$ of $P \xrightarrow{\pi} \mathcal{M}$ is broken down to $\mathbb{Z}_3 \times \mathrm{E}_6 \times \mathrm{E}_8$, however, in the physical literature it is often regarded as just $\mathrm{E}_6 \times \mathrm{E}_8$. Of course, the same procedure can be carried out for the SO(32)-gauge bundle, where the gauge group is broken to $\mathrm{SO}(26) \times \mathrm{U}(1)$, but this will not be considered further. This is called the **standard embedding** and is usually referred to as the **embedding of the spin connection** in the physical literature. However, as it was shown, the notion of "embedding the spin connection in the gauge connection" is a rather unfortunate



(Image of the manifold by Andrew J. Hanson.)

FIGURE 8.3. Calabi-Yau compactification of the heterotic string.

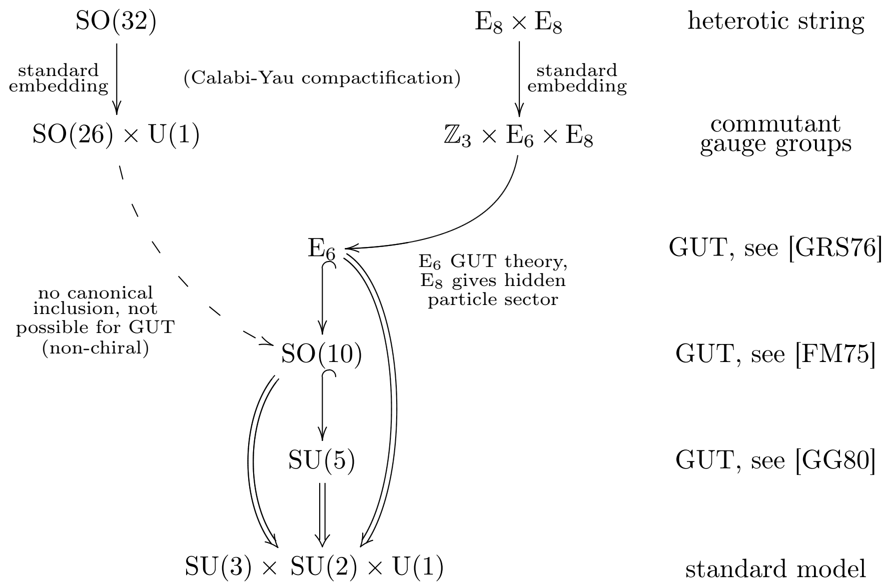


FIGURE 8.4. Inclusions of several GUT gauge groups with non-exotic fermions, as the SO(10)-GUT model does not contain any additional fermions besides the Standard Model fermions and the right-handed neutrino. It also unifies each generation into a single irreducible representation. A brief summary of those GUT models is found in sec. 10.1.

misconception from the mathematical point of view, as it is the holonomy bundle which gets embedded into the gauge bundle.

8.8. Calabi-Yau compactification of heterotic strings

The constructions of the previous sections can be summarized as follows, cf. [DEF+99, p. 1112]: Given a Calabi-Yau threefold (K^6, J, g_0), there is a classical solution for 10d supergravity (coupled via Chern-Simons terms to super Yang-Mills theory), such that the following properties are achieved:

- The space-time metric on $\mathcal{M}^{10} = \mathbb{R}^4 \times K^6$ has the product form $G = \eta \times g_0$.
- The axion 3-form field $\tilde{H}$ vanishes.
- The dilaton field Φ is constant.
- The fermionic superpartner fields (gravitino χ_μ, dilatino φ and gaugino λ) vanish.
- The gauge field is obtained by embedding the spin connection as a gauge connection, such that the gauge symmetry is broken:

$$\begin{array}{rcl} \mathrm{E}_8 \times \mathrm{E}_8 & \longrightarrow & \mathbb{Z}_3 \times \mathrm{E}_6 \times \mathrm{E}_8 \\ \mathrm{SO}(32) & \longrightarrow & \mathrm{SO}(26) \times \mathrm{U}(1) \end{array} \qquad \text{(via standard embedding)}$$

- $\mathcal{N} = 1$ 4d super-Poincaré symmetry.

The remaining commutant gauge group $\mathrm{SO}(26)\times\mathrm{U}(1)$ of the Calabi-Yau compactified heterotic SO(32)-string theory can be shown to yield only non-chiral representations. Thus, it is not interesting for any kind of (semi-)realistic theories of of elementary particles.

For the heterotic $\mathrm{E}_8 \times \mathrm{E}_8$-string a totally different picture is revealed. Under the maximal subgroup $\mathrm{SU}(3) \times \mathrm{E}_6 \subset \mathrm{E}_8$ used in the standard embedding into the first copy of E_8, the fundamental (or equivalently the adjoint) representation 248 of E_8 decomposes as

$$\begin{aligned} \mathrm{E}_8 &\longrightarrow \mathrm{SU}(3)\times\mathrm{E}_6 \\ 248 &\mapsto (1,78) + \underbrace{(3,27)+(\overline{3},\overline{27})}_{\text{suitable for } \mathrm{E}_6\text{-GUT}} + (8,1) \end{aligned} \qquad \text{where} \qquad \begin{array}{rl} 1 & \text{trivial representation} \\ 3,\overline{3} & \text{fund./conj. rep. of SU(3)} \\ 27,\overline{27} & \text{fund./conj. rep. of } \mathrm{E}_6 \\ 8 & \text{adjoint rep. of SU(3)} \\ 78 & \text{adjoint rep. of } \mathrm{E}_6 \end{array}$$

The 27-dimensional fundamental and conjugate representations of E_6 are both suitable for a grand unified theory (GUT) with gauge group E_6, which is known to be related to the rather realistic SO(10)-GUT. Along with smaller GUT gauge groups, the standard model is (at least in principle) obtained along the inclusions indicated in fig. 8.4. The second, unbroken copy of E_8 found in the remaining gauge symmetry is effectively removed in the compactified theory—it forms a **hidden particle sector**, which interacts with the "realistic" particles of the E_6-GUT only via the gravitation and other universal interactions.

Let n_{L} be the number of massless left-handed Weyl fermions transforming under the fundamental representation 27, and likewise n_{R} the number for R-handed Weyl fermions transforming under $\overline{27}$. For massless modes of opposite chirality, there is a simple mechanism to combine and form massive fermions, i.e. only the difference $|n_{\mathrm{L}} - n_{\mathrm{R}}|$ gives the number of **massless fermion generations**. This number is provided by the algebraic index of the internal manifold's Dirac operator $\not{D}_6$, which is given by the topology of the internal manifold in terms of the Euler characteristic:

$$n_{\mathrm{L}} - n_{\mathrm{R}} = \mathrm{index}(\not{D}_6) = -\frac{\chi(K^6)}{2}.$$

In this formula one essentially regards n_{L} and n_{R} as the kernels of a chirality-splitted Dirac operator (since any massless fermion field is an element of the kernel of the Dirac operator), see [LM89, II.§6] for the mathematical details. Since experiment shows the existence of three chiral fermion generations, only Calabi-Yau manifolds with Euler characteristic $\chi(K^6) = \pm 6$ should be considered in order to achieve semi-realistic compactifications. This is explained in [Ura03, §12.3.2] in more detail.

Of course, following the same Kaluza-Klein mechanism as for the 6-torus, the Calabi-Yau compactification yields a number of massless scalars, i.e. moduli fields. Given a internal Calabi-Yau manifold K^6, the relevant information on the topology is encoded in the Hodge numbers $h^{p,q}$. Due to various symmetries, $h^{1,1}$ and $h^{2,1}$ are the only two independent topology numbers, and the Euler characteristic can be expressed as

$$\chi(K^6) = 2\left(h^{1,1} - h^{2,1}\right).$$

In order to get three chiral fermion generations in the effective theory, the Hodge numbers should satisfy $|h^{1,1} - h^{2,1}| = 3$. Moreover, it can be shown (see [BBS07, §9.6]) that for a given Calabi-Yau topology, the corresponding moduli space locally has the product structure

$$\mathcal{M}^{\mathrm{CY}}_{\mathrm{het}}(K^6) = \underbrace{\mathcal{M}_J(K^6)}_{\text{complex-structure moduli space}} \times \overbrace{\mathcal{M}_\omega(K^6)}^{\text{Kähler-structure moduli space}}.$$

The **Kähler-structure moduli space** $\mathcal{M}_\omega(K^6)$ is parameterized by $h^{1,1}$ moduli fields, whereas the **complex-structure moduli space** is $h^{2,1}$-dimensional. From the physical point of view, the corresponding $h^{1,1} + h^{2,1}$ massless scalars are contained in $\mathcal{N} = 1$ chiral supermultiplets. Again, the moduli fields give rise to an additional interaction which is not observed, see chap. 11 for a short account on this problem.

It should be noted, that Calabi-Yau compactifications can also be carried out for type-II superstrings in a consistent way, which was first considered in [CHSW85]. In that case, one can construct either 10d $\mathcal{N} = 1$ or $\mathcal{N} = 2$ superstring theories, however the lack of closed-string Yang-Mills states is not resolved.

8.9. Problems of the smooth compactifications

Both the toroidal compactification of the 10d heterotic string and its Calabi-Yau compactification have certain benefits, but also also a number of drawbacks:

- As it was shown in sec. 8.4, the **toroidal compactification** does not break any supersymmetry and thus yields a 4d $\mathcal{N} = 4$ supersymmetric theory upon compactification—which is non-chiral and therefore highly unrealistic. However, since a 6-torus is a very simple manifold with an ample explicit description, actual calculations can be carried out without much problems.
- **Calabi-Yau compactifications** on the other hand provide a beautiful correspondence between (potentially) realistic GUT theories and the internal space-time. Most results only depend on the topology of the internal manifold and the explicit description of the moduli spaces. However, not a single explicit Calabi-Yau metric (with the exception of the trivial 6-torus) is known to this day—so there are no results directly depending on the geometry, as the field equations cannot be solved.
- It should be pointed out, that there is an alternative to compactification presented by the **Randall-Sundrum model**, see [RS99]. It essentially relies on choosing a suitable 3+1-dimensional semi-Riemannian submanifold in the 10d bulk space. However, this will not be investigated any further.

To summarize, a simply-calculable but highly unrealistic model was found (toroidal compactification), whereas the other model with many physically interesting properties (Calabi-Yau compactification) is constructed from such abstract notions, that actual calculations are impossible.

A few years after the Calabi-Yau compactification was first considered, the idea was put forward to allow for internal spaces with certain singularities. In the next chapter it is shown, how those so-called "orbifolds" can be used to construct calculable compactification models with 4d $\mathcal{N} = 1$ supersymmetry.

CHAPTER 9

Toroidal Orbifold Model Building

The problems mentioned at the end of the preceding chapter are dealt with by introducing orbifolds—spaces which can contain singularities, where the curvature or other typical structures may be infinite. After a general mathematical introduction to orbifolds, the notion of holonomy is introduced in this context. This allows to attack the problem of 4d $\mathcal{N}=1$ SUSY in the same fashion as it was done in the case of Calabi-Yau manifolds. The particular class of toroidal orbifolds with SU(3)-holonomy allows for a successful compactification of the heterotic string, which can be carried out in explicit terms due to the underlying torus structure. Many details are omitted, however, the issues of anomaly cancellation and SUSY breaking are discussed in a rather lengthy manner. An essential point in orbifold compactifications is the appearance of a complex phase under the action of the geometry-defining point group. Only the invariant states—those with trivial phase—survive the compactification process, which is also responsible for the reduction to 4d $\mathcal{N}=1$ SUSY. In the last section the specified form of the Hilbert spaces is investigated. All the preliminaries for the $\mathbb{Z}_6$-II orbifold model discussed in the next chapter are provided.

9.1. General orbifolds

In his study of hyperbolic differential equations, Thurston has introduced the notion of an orbifold in order to deal with singularities usually found in the solutions to such equations. Orbifolds and manifolds are defined in a similar manner—however, since orbifolds also allow for presence of singularities, the notion of a manifold is generalized. Conversely, manifolds can be regarded as the special case of orbifolds not containing any singularities.

A n-dimensional **topological orbifold** O is a topological second-countable Hausdorff space X_O, called the **underlying space**, together with an orbifold structure, which is specified by an orbifold atlas defined as follows: An **orbifold chart** for $U \subset X_O$ is a continuous map $\phi : V \longrightarrow U$ for $V \subset \mathbb{R}^n$, such that there is a finite group G acting on V by linear transformations and a homeomorphism $\psi : V/G \xrightarrow{\approx} U$ satisfying $\phi = \psi \circ \pi$ for $\pi : V \twoheadrightarrow V/G$ being the quotient projection mapping. This can be shown in diagrammatic form as follows:

$$\begin{tikzcd}
\text{orbifold} & U & & V \arrow[ll, "\text{chart } \phi"', "\text{(continuous fctn.)}"] \arrow[d, two heads, "\pi"', "\substack{\text{quotient}\\ \text{projection}}"] \arrow[dl, phantom, "\circlearrowright"] & \text{flat space} \\
& & & V/G \arrow[ull, bend left, "\psi", "\approx"'] & \text{prototype space}
\end{tikzcd}$$

The reader might object to the used terminology, that one would rather call ψ instead of ϕ the "chart mapping". However, due to the singular nature of the prototype space V/G it is much more practical to construct a G-invariant mapping $\phi : V \longrightarrow U$, which then descents to a homeomorphism ψ. Also note that the direction of the chart mapping is reversed compared to the manifold case, i.e. it goes from the flat (or prototype) space to the orbifold. This is of course irrelevant for the construction, as the mapping ψ is a homeomorphism.

Two orbifold charts $\phi_\alpha : V_\alpha \longrightarrow U_\alpha$ and $\phi_\beta : V_\beta \longrightarrow U_\beta$ with nonempty overlap $U_\alpha \cap U_\beta$ are called compatible, if for $\phi_\alpha(p) = \phi_\beta(q)$ there exist neighborhoods $p \in V_p \subset V_\alpha$ and $q \in V_q \subset V_\beta$ as well as a homeomorphism $h_{\alpha\beta} : V_p \xrightarrow{\approx} V_q$, such that $\phi_\alpha = \phi_\beta \circ h_{\alpha\beta}$ is satisfied. Again, this rather technical statement can be made more accessible in a diagram:

$$\begin{tikzcd}
V_\alpha \arrow[d, two heads, "\pi_\alpha"'] \arrow[rrr, bend right, "\phi_\alpha"'] & \supset & V_p \arrow[r, "\phi_\alpha|_{V_p}", "\text{(cont. fctn.)}"'] \arrow[rr, bend left=50, "\text{transition mapping } h_{\alpha\beta}", "\approx \; \circlearrowleft"'] & U & V_q \arrow[l, "\phi_\beta|_{V_q}"', "\text{(cont. fctn.)}"] & \subset & V_\beta \arrow[d, two heads, "\pi_\beta"] \arrow[lll, bend left, "\phi_\beta"] \\
V_\alpha/G \arrow[rrru, "\underset{\psi_\alpha}{\approx}"] & \supset & V_p/G \arrow[rr, dashed, "\approx", "\text{induced mapping } h_{\alpha\beta}/G"'] & & V_q/G & \subset & V_\beta/G \arrow[lllu, "\underset{\psi_\beta}{\approx}"']
\end{tikzcd}$$

A collection $\{\phi_\alpha : V_\alpha \longrightarrow U_\alpha\}$ of compatible orbifold charts with $\bigcup_\alpha U_\alpha = X_O$ is called an **orbifold atlas** or (topological) **orbifold structure**. Furthermore, if all orbifold chart transition functions $h_{\alpha\beta}$ are diffeomorphisms (instead of homeomorphisms), this gives rise to the notion of a **differentiable orbifold**. Likewise, a **complex orbifold** is defined using biholomorphic orbifold chart transition mappings $h_{\alpha\beta}$.

Since orbifolds are locally compact and locally path connected it follows that an orbifold is connected if and only if it is path-connected. Thus, just like for manifolds one does not need to distinguish between those notions from elementary topology.

It follows from the definition, that certain orbifolds are quotient spaces M/G of a discrete group G acting on a manifold M, see [Thu81, propos. 13.2.1] for a full mathematical proof.[a] However, there are orbifolds which cannot be written in that fashion, e.g. weighted projective spaces, see [Joy00, p. 134]. For the rest of the chapter only quotient space orbifolds are considered, particularly complex orbifolds constructed from discrete groups holomorphically acting on a complex manifold. In the context of string theory compactification only **toroidal orbifolds** are considered, i.e. quotient orbifolds $O = T^n/G$, where T^n is the n-torus.

The geometrical properties of orbifolds are very rich. From the mathematical point of view, a particular important question is whether a given orbifold can be written as a quotient space of a discrete group acting on a manifold. Using general methods of algebraic geometry, one can "blow up" and "resolve" certain types of orbifold singularities—however, in general this is a complicated issue with almost no explicit description. Fortunately, for toroidal orbifolds there exists an ample description in terms of toric varieties, see [Ful93] for a general introduction to the subject. Physical questions in this context will be discussed later.

9.2. Holonomy of quotient orbifolds

Given an oriented manifold M and a faithful, orientation-preserving action of the finite group G on M, one can define for each $p \in M$ the **stabilizer subgroup**

$$\operatorname{stab}(p) := \big\{g \in G : g \cdot p = p\big\},$$

consisting of those group elements that leave the point p invariant. If $\operatorname{stab}(p) = \{e\}$ the point $p \in M$ is called a **non-singular point**. Conversely, the **set of singular points** is defined as

$$S := \big\{[p] \in M/G : p \in M \text{ and } g \cdot p = p \text{ for some } e \neq g \in G\big\}.$$

By means of the Newman-Thurston theorem, S is a closed set with empty interior. Furthermore, the singular set is not necessarily a submanifold and may consist of several connection components. [Bor92] provides a very readable mathematical introduction to orbifolds.

The definition of a **real** or **complex orbifold** from the previous section is then equivalent to requiring any singular point $s \in M/G$ to be locally isomorphic to a prototype quotient

[a]Quotient orbifolds $O = M/G$ are called **good**, whereas **bad** orbifolds cannot be written as quotient spaces. This stresses the fact, that the quotient space description simplifies many problems. For example, a teardrop is an example of a bad orbifold, where the underlying space is simply S^2 with a single singularity.

singularity $\mathbb{R}^m/G$ or $\mathbb{C}^m/G$ for finite subgroups $G \subset \mathrm{GL}(m;\mathbb{R})$ or $\mathrm{GL}(m;\mathbb{C})$. Note that any complex orbifold can be regarded as a real orbifold in a natural way due to $\mathrm{GL}(m;\mathbb{C}) \subset \mathrm{GL}(2m;\mathbb{R})$.

It should be stressed that the set of singular points by no means is restricted to isolated 0-dimensional point-like singularities. For example, consider the 3-dimensional spatial rotations in $\mathrm{SO}(3)$. After fixing a line L through the origin, the stabilizer subgroup consists of rotations $\mathrm{SO}(2) \subset \mathrm{SO}(3)$, which keep this line L invariant. Now embed the cyclic group $\mathbb{Z}_n$ into $\mathrm{SO}(2)$ by regarding the elements as rotations by $\frac{2\pi}{n}k$ radiants for $k = 0, \ldots, n-1$. The group structure is then obvious, and by this induced discrete rotation action $\mathbb{R}^3/\mathbb{Z}_n$ has at least the entire (1-dimensional) line L in the singularity set. In the physical context, however, only 0-dimensional point singularities and 2-dimensional torus singularities will be of interest.

Obviously, by removing the singular points S from any quotient orbifold M/G, an ordinary manifold $(M/G) \setminus S$ of dimension

$$\dim(M/G) \setminus S = \dim M$$

is obtained. A (real) quotient orbifold $O = M/G$ can be equipped with a **Riemannian metric** g, which reduces to an ordinary Riemannian metric on the non-singular manifold part, if it can be identified with an G-invariant Riemannian metric on $\mathbb{R}^m/G$ wherever O is locally isomorphic to $\mathbb{R}^m/G$. Likewise, a **Kähler metric** g on a complex quotient orbifold reduces to an ordinary Kähler metric on the non-singular part $O \setminus S$. This gives rise to the notions of **Riemannian** and **Kähler orbifolds**.

In particular, the Calabi conjecture also holds for compact Kähler orbifolds—however, it must be clarified what the first Chern class of an orbifold means. In a more general context, one can define a G-equivariant holonomy and corresponding G-equivariant characteristic classes for orbifold quotient spaces M/G, see [FV87, §4] for a short introduction to G-equivariant geometry. Fortunately, those rather lengthy constructions are not needed for the case at hand. From a deeper understanding of characteristic classes using Chern-Weil theory, it follows that the first Chern class indeed only depends on the determinant line bundle $K_M \xrightarrow{\pi} M$. The determinant line bundle is well-defined for $O = M/G$ provided G is a discrete subgroup of $\mathrm{SL}(m;\mathbb{C})$ instead of $\mathrm{GL}(m;\mathbb{C})$. Thus, the first Chern class $c_1(O) \in H^n(O;\mathbb{R})$ is well-defined for complex quotient orbifolds $O = M/G$ with $G \subset \mathrm{SL}(m;\mathbb{C})$ being a discrete subgroup. The orbifold variant of the Calabi conjecture then states that for any compact Kähler orbifold O with $c_1(O) = 0$ there exists an unique Ricci-flat Kähler metric in any Kähler class $[\omega]$ on O.

For real and complex quotient orbifolds $O = M/G$ with a Riemannian or Kähler metric g the **holonomy group** $\mathrm{Hol}(g)$ is defined to be the holonomy group of the non-singular manifold part $O \setminus S$, i.e.

$$\mathrm{Hol}^{\mathrm{orbifold}}(g) := \mathrm{Hol}^{\mathrm{manifold}}(g|_{O\setminus S}).$$

Just like for Kähler manifolds, $\mathrm{Hol}(g) \subseteq \mathrm{U}(m)$ holds for any Kähler orbifold O. Furthermore, a **Calabi-Yau orbifold** is a complex Kähler orbifold O with a Kähler metric g, such that $\mathrm{Hol}(O) = \mathrm{SU}(m)$. In particular, given a Kähler manifold M and a finite group G holomorphically acting on M, then M/G is a Kähler orbifold. Likewise, for a Calabi-Yau manifold

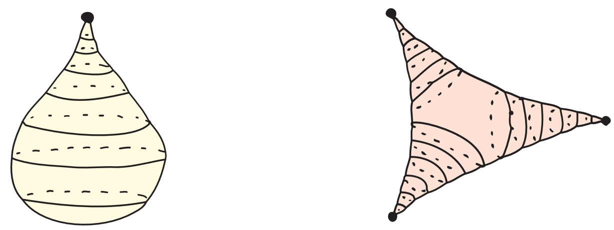


FIGURE 9.1. Two orbifolds with singular points: a teardrop and the toroidal $T^2/\mathbb{Z}_3$ orbifold. Except for the singularities, an orbifold is a manifold.

M equipped with a finite group G of holomorphic isometries of M, the quotient space M/G yields a Calabi-Yau orbifold, provided there exists a G-invariant holomorphic volume form θ on the Calabi-Yau manifold M.

Thus, quite a number of important results obtained for smooth manifolds in chap. 5 extend to singular orbifolds without much problem. The 6-dimensional Calabi-Yau orbifolds are of particular importance in the context of string theory compactification, as they potentially allow for $\mathcal{N} = 1$, albeit along a different reasoning as the one discussed in the previous chapter for Calabi-Yau manifolds.

9.3. Toroidal orbifolds

Consider a 2-dimensional torus $T^2 = S^1 \times S^1$, whose geometrical properties are fully specified by modding out a 2-dimensional lattice $\Lambda_2 = \{\mathbb{Z}e_1 + \mathbb{Z}e_2\} \subset \mathbb{R}^2$ from $\mathbb{R}^2$, where e_1 and e_2 are two linearly independent vectors spanning the lattice, i.e.

$$T^2_\Lambda \cong \mathbb{R}^2/\Lambda_2.$$

This was already done in sec. 7.5. Due to $\mathbb{R}^2 \cong \mathbb{C}$ as vector spaces, the same geometry can be described by choosing a non-zero complex number $\rho \in \mathbb{C}^\times$, such that the splitting $\rho = e_1 + \mathrm{i}e_2$ provides the two basis vectors. It is then more appropriate to formulate

$$T^2_\Lambda \cong \mathbb{C}/\Lambda_\rho,$$

which stresses the fact, that a flat torus T^2 has in fact an underlying complex structure—effectively turning T^2 into a compact flat Kähler manifold, called the **flat Kähler torus**. In the same fashion any compact even-dimensional torus T^{2n} can be regarded as $\mathbb{C}^n/\Lambda$, i.e. a flat Kähler manifold. Moreover, for the flat Kähler torus there exists a holomorphic volume form ω on $T^{2n} = \mathbb{C}^n/\Lambda$.

Now let P be a finite group of T^{2n}_Λ-automorphisms preserving the Kähler metric g, the complex structure J and the holomorphic volume form ω, then $O := T^{2n}_\Lambda/P$ is a compact Kähler orbifold with $P \subset \mathrm{SL}(n;\mathbb{C})$. This is equivalent to requiring P to act crystallographically on the torus lattice, i.e. P contains isometric automorphisms of Λ. In this context P is called the **point group** which defines the corresponding **toroidal orbifold**, and the elements $\theta \in P$ are called **twists**. One introduces the **space group** $S := \Lambda \ltimes P$, acting on a vector $x \in \mathbb{R}^{2n}$ via

$$(\ell, \theta) \cdot x := \theta x + \ell$$

for $\theta \in P$ and $\ell \in \Lambda$. The space group S can be regarded as a group of isometric automorphisms of the torus lattice, i.e. $S \subset \mathrm{Aut}(\Lambda)$. The group structure is defined such that associativity $\big[(\ell_1, \theta_1)(\ell_2, \theta_2)\big] \cdot x = (\ell_1, \theta_1) \cdot \big[(\ell_2, \theta_2) \cdot x\big]$ holds, which implies

$$\begin{aligned}
&(\ell_1, \theta_1)(\ell_2, \theta_2) = (\theta_1\ell_2 + \ell_1, \theta_1\theta_2)\\
&(\ell, \theta)^{-1} = (-\theta^{-1}\ell, \theta^{-1})\\
&\mathrm{Ad}_{(\ell,\theta)}(\tilde{\ell}, 1) = (\ell, \theta)(\tilde{\ell}, 1)(\ell, \theta)^{-1} = (\theta\tilde{\ell}, 1)
\end{aligned}$$

as usual for the semidirect product. The toroidal orbifold O may then be written as

$$O = T^{2n}_\Lambda/P \cong \mathbb{R}^{2n}/S.$$

The description of toroidal orbifolds in terms of a $\mathbb{R}^{2n}$ (or equivalently a $\mathbb{C}^n$) quotient space makes the explicit description particularly simple, as one just needs to consider S-invariant objects on $\mathbb{R}^{2n}$ (or $\mathbb{C}^n$), e.g. S-invariant mappings $\mathbb{C}^n \longrightarrow \mathbb{R}$.

Let $S = \Lambda \ltimes P$ be a space group and consider two points $x, y \in \mathbb{R}^{2n}$ such that the relation $x = (\ell, \theta) \cdot y = \theta y + \ell$ holds for some $(\ell, \theta) \in \Lambda \ltimes P = S$. On the corresponding toroidal orbifold $O = \mathbb{R}^{2n}/S \cong T^{2n}_\Lambda/P$ those points are not only identified, but also the tangent vectors $v \in \mathrm{T}_x\mathbb{R}^n$ are identified with the tangent vectors $w \in \mathrm{T}_y\mathbb{R}^n$ rotated by θ. Now consider a path γ in $\mathbb{R}^n$ from x to y. On the orbifold O this reduces to a closed loop, such that by parallel transporting a vector $v \in \mathrm{T}_xO$ along γ it remains unchanged since the torus is flat, but differs

point group	**generator v**
$\mathbb{Z}_3$	$\frac{1}{3}(1,1,-2)$
$\mathbb{Z}_4$	$\frac{1}{4}(1,1,-2)$
$\mathbb{Z}_6$-I	$\frac{1}{6}(1,1,-2)$
$\mathbb{Z}_6$-II	$\frac{1}{6}(1,2,-3)$
$\mathbb{Z}_7$	$\frac{1}{7}(1,2,-3)$
$\mathbb{Z}_8$-I	$\frac{1}{8}(1,2,-3)$
$\mathbb{Z}_8$-II	$\frac{1}{8}(1,3,-4)$
$\mathbb{Z}_{12}$-I	$\frac{1}{12}(1,4,-5)$
$\mathbb{Z}_{12}$-II	$\frac{1}{12}(1,5,-6)$

point group	**generator v**	**generator w**
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\frac{1}{2}(1,0,-1)$	$\frac{1}{2}(0,1,-1)$
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\frac{1}{3}(1,0,-1)$	$\frac{1}{3}(0,1,-1)$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\frac{1}{2}(1,0,-1)$	$\frac{1}{4}(0,1,-1)$
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\frac{1}{4}(1,0,-1)$	$\frac{1}{4}(0,1,-1)$
$\mathbb{Z}_2 \times \mathbb{Z}_6$-I	$\frac{1}{2}(1,0,-1)$	$\frac{1}{6}(0,1,-1)$
$\mathbb{Z}_2 \times \mathbb{Z}_6$-II	$\frac{1}{2}(1,0,-1)$	$\frac{1}{6}(1,1,-2)$
$\mathbb{Z}_3 \times \mathbb{Z}_6$	$\frac{1}{3}(1,0,-1)$	$\frac{1}{6}(0,1,-1)$
$\mathbb{Z}_6 \times \mathbb{Z}_6$	$\frac{1}{6}(1,0,-1)$	$\frac{1}{6}(0,1,-1)$

Table 9.1. Suitable point groups $\mathbb{Z}_n \subset \mathrm{SU}(3)$ and $\mathbb{Z}_n \times \mathbb{Z}_m \subset \mathrm{SU}(3)$ for toroidal Calabi-Yau orbifolds with "generators" $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$. The lists are found with additional information in [KKK+90, tbl. 1] for the $\mathbb{Z}_n$ case and in [FIQ89, tbl. 1] for $\mathbb{Z}_n \times \mathbb{Z}_m$-orbifolds.

from the original vector by a rotation θ. Thus, the holonomy group of toroidal orbifolds is a discrete group in contrast to the holonomy Lie groups encountered so far. Furthermore, since $\mathrm{Hol}(O) \cong P$ this makes the construction of orbifolds with specific holonomy particularly simple.

9.4. Six-dimensional toroidal Calabi-Yau orbifolds

In order to arrive at an 4d $\mathcal{N} = 1$ supersymmetric effective field theory, one considers (complex) 3-dimensional Calabi-Yau orbifolds with holonomy group SU(3). This allows for a global covariantly constant (parallel) spinor generating the supersymmetry as outlined in the previous chapter—however, the actual SUSY breaking in the 4d theory happens in a different manner, see sec. 9.11. For manifolds the requirement of SU(3)-holonomy leads to the practical problem that no explicit description of any non-trivial Calabi-Yau metric is known. On the other hand, for toroidal orbifolds the metric is flat and the non-trivial holonomy curves are located around the singularities.

Fortunately, both approaches are compatible, such that 6-dimensional toroidal Calabi-Yau orbifolds are the most natural type of compactification space after the "failures" discussed in the last chapter. In the physical context, orbifold compactification essentially combines the phenomenological success of Calabi-Yau compactification (breaking of gauge symmetry, 4d $\mathcal{N} = 1$ SUSY) with the calculability of toroidal compactification. Furthermore, in the particular case of 6-dimensional toroidal orbifolds $T^6/P \cong \mathbb{R}^6/S$, the singularities—which arise from fixpoints of the point group's torus automorphisms—can be dealt with using "crepant resolutions" provided the orbifold's fundamental group $\pi_1(T^6/S)$ is finite, see [Joy00, §6.6] or [Joy07, §7.5].

For a suitable choice of real coordinates $X^1, \ldots, X^6$ for $\mathbb{R}^6$, the T^6-torus lattice is simply $\Lambda = \mathbb{Z}^6$. However, in physics one usually considers lattices spanned by the simple roots of certain Lie groups, see the references mentioned in tab. 9.1 and the root lattices in app. B. One can also consider the complex coordinates $Z^1, \ldots, Z^3$ via the identification

$$Z^a = \left(X^{2a-1} + \mathrm{i}X^{2a}\right). \tag{9.1}$$

In order to act crystallographically on the lattice Λ, the point group P has to be a discrete subgroup of SO(6). If attention is restricted to abelian point groups, it further belongs to the abelian subgroup of SO(6), which is fully described by its Cartan algebra. Let M^{12}, M^{34}, M^{56} denote the three generators of this subalgebra (with action on the respective pairs of real coordinates), then—utilizing the complex coordinates—the point group elements $\theta \in P$ act diagonally and can be written as

$$\theta^k = \mathrm{e}^{2\pi \mathrm{i} k(v_1 M^{12} + v_2 M^{34} + v_3 M^{56})} \qquad \rightsquigarrow \qquad \theta^k Z := \begin{pmatrix} \mathrm{e}^{2\pi \mathrm{i} k v_1} Z^1 \\ \mathrm{e}^{2\pi \mathrm{i} k v_2} Z^2 \\ \mathrm{e}^{2\pi \mathrm{i} k v_3} Z^3 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{2\pi \mathrm{i} \eta_1^k} Z^1 \\ \mathrm{e}^{2\pi \mathrm{i} \eta_2^k} Z^2 \\ \mathrm{e}^{2\pi \mathrm{i} \eta_3^k} Z^3 \end{pmatrix}$$

for $0 \leq |v_a| < 1$ and $\eta_a^k := k v_a \bmod 1$, such that $0 \leq \eta_a^k < 1$. In terms of the original real coordinates $X^1, \ldots, X^6$ this is equivalent to a block diagonal matrix of rotations, i.e.

$$\theta^k = \left(\begin{array}{c|c|c} \vartheta_1^k & 0 & 0 \\ \hline 0 & \vartheta_2^k & 0 \\ \hline 0 & 0 & \vartheta_3^k \end{array}\right) \qquad \text{with} \qquad \vartheta_a^k := \begin{pmatrix} \cos(2\pi\eta_a^k) & -\sin(2\pi\eta_a^k) \\ \sin(2\pi\eta_a^k) & \cos(2\pi\eta_a^k) \end{pmatrix}$$

for $a = 1, 2, 3$. Finally, the restriction to point groups $P \subset \mathrm{SU}(3)$ for $\mathcal{N} = 1$ SUSY imposes the further condition

$$v_1 + v_2 + v_3 = 0$$

on the possible generator parameters, see sec. B.5. Nevertheless, all possible point groups are classified (see [KKK+90, §2] and [FIQ89]) in two categories:

(1) **Cyclic groups** $\mathbb{Z}_N = \{\theta^k : k = 0, \ldots, n-1\}$ for $N = 3, 4, 6, 7, 8, 12$ are fully described by specifying the generator θ. In the cases $N = 6, 8, 12$ there are two different embeddings of the cyclic groups $\mathbb{Z}_N$ into $\mathrm{SU}(3) \subset \mathrm{SO}(6)$, leading to two possible choices of the generator θ.
(2) **Double-cyclic groups** $\mathbb{Z}_N \times \mathbb{Z}_M = \big\{(\theta_1^k, \theta_2^l) : k = 0, \ldots, N-1; l = 0, \ldots, M-1\big\}$ have two independent generators θ_1 and θ_2. Again, there might be different embeddings into $\mathrm{SU}(3) \subset \mathrm{SO}(6)$.

All possible finite point groups $\mathbb{Z}_N$ or $\mathbb{Z}_N \times \mathbb{Z}_M$ suitable for orbifold compactifications of the heterotic string are listed in tab. 9.1, yielding 6-dimensional toroidal Calabi-Yau orbifolds.[b]

9.5. Example with 0d conical singularities: $T^6/\mathbb{Z}_3$-orbifold

A simple example of an toroidal Calabi-Yau orbifold is gained by identifying out the discrete rotation group $\mathbb{Z}_3$ from the 6-torus. This was also the example first considered in the context of string compactifications, see [DHVW85]. The three complex unit roots $\sqrt[3]{1} \subset \mathrm{U}(1)$ provide an explicit representation of the group $\mathbb{Z}_3 = \{\theta^k : k = 0, 1, 2\}$ with the generator

$$\theta := {}_1\sqrt[3]{1} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i},$$

which acts on the complex coordinates by complex multiplication. Let $T^6 = T^2 \times T^2 \times T^2$ be the flat Kähler 6-torus and $Z^1, \ldots, Z^3$ its complex coordinates as in the previous section. The torus lattice is chosen to be the root lattice of $\mathrm{SU}(3)^3$, see sec. B.5. The root system of each SU(3) contains 6 roots which span an 2-dimensional Euclidean space, as required—see sec. B.5 for the full story. The six simple roots (two for each SU(3)-copy) are

$$\begin{aligned} \alpha_1^a &= \sqrt{2} \\ \alpha_2^a &= \sqrt{2}\theta \end{aligned} \qquad \text{for } a = 1, 2, 3,$$

[b]Note that some sources (e.g. [KKK+90]) use the name **Coxeter orbifolds**, which stresses the used algebraic approach to the classification of possible point groups. However, the name "toroidal Calabi-Yau orbifold" emphazises the SU(3)-holonomy used for $\mathcal{N} = 1$ SUSY breaking and their construction from 6-tori.

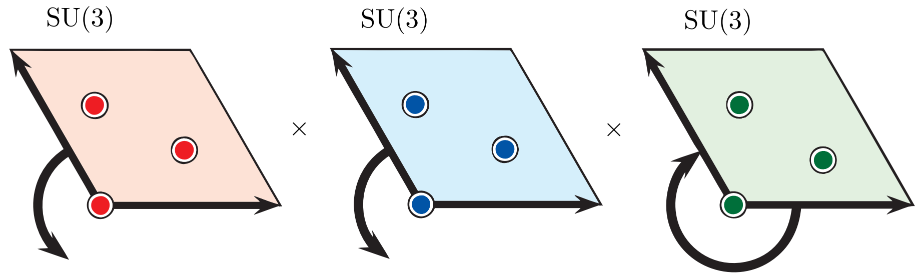


FIGURE 9.2. The SU(3) × SU(3) × SU(3) root lattice with fixpoints as used in the definition of the $\mathbb{Z}_3$ orbifold.

which span the 6d torus lattice $\Lambda_{\mathrm{SU}(3)^3}$. As before, the action of the point group $\mathbb{Z}_3$ on the torus $T^6_{\mathrm{SU}(3)^3}$ is defined by a mapping

$$\begin{aligned} \theta^k : T^6_{\mathrm{SU}(3)^3} &\longrightarrow T^6_{\mathrm{SU}(3)^3} \\ Z^a &\mapsto \mathrm{e}^{2\pi \mathrm{i} \eta^k_a} Z^a \end{aligned},$$

where $v = (v_1, v_2, v_3) = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ is the generating vector, cf. tab. 9.1. This amounts to counter-clockwise 120° rotations in the first two SU(3)-planes and a clockwise 240° rotation in the third SU(3)-plane, see fig. 9.2. Obviously, this provides an explicit representation of the group $\mathbb{Z}_3$ on the $T^6_{\mathrm{SU}(3)^3}$-torus. From sec. 9.3 it follows that $T^6_{\mathrm{SU}(3)^3}/\mathbb{Z}_3$ is a toroidal Calabi-Yau orbifold, see [Joy00, ex. 6.6.3] for a more mathematical proof.

It remains to determine the number of singularities, which is equivalent to the number of fixpoints under the action of the point group $\mathbb{Z}_3$. Since all three complex coordinates are mutually orthogonal, one simply has to solve the fixpoint equation $Z^a_{\mathrm{fp}} = \theta^k Z^a_{\mathrm{fp}} + \ell$ for $\ell \in \Lambda_{\mathrm{SU}(3)}$ for a single SU(3)-plane $a = 1, 2$ or 3 since all are equivalent, i.e.

$$(\mathrm{Id} - \theta^k) Z^a_{\mathrm{fp}} \in \Lambda_{\mathrm{SU}(3)} \iff (1 - \mathrm{e}^{2\pi \mathrm{i} \eta^k_a}) Z^a_{\mathrm{fp}} = \sum_{l=1}^{2} n^a_l \alpha^a_l \quad \text{for } a = 1, 2, 3, \tag{9.2}$$

where $n^a_l \in \mathbb{Z}$ are the multiples of the simple roots α^a_l that span the $\mathrm{SU}(3)^3$-lattice. This is called the **lattice shift vector** and is an individual quantity for each fixpoint. For $k = 1$ this equation then yields 3 fixpoints

$$\begin{aligned} Z^a_{\mathrm{fp},1} &= 0 && \text{with lattice shift vector } n^a = (0,0) \\ Z^a_{\mathrm{fp},2} &= \sqrt{\tfrac{2}{3}}\mathrm{i} && \text{with lattice shift vector } n^a = (1,1) \\ Z^a_{\mathrm{fp},3} &= \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{6}}\mathrm{i} && \text{with lattice shift vector } n^a = (1,0) \end{aligned}$$

within each of the independent complex coordinates Z^a for $a = 1, 2, 3$, as shown in fig. 9.2. Thus, each of the three T^2-tori has three fixpoints and can be imagined as a "pillow" with three tips, see fig. 9.1. The action of the point group $\mathbb{Z}_3$ in total has $3 \cdot 3 \cdot 3 = 27$ fixpoints on the 6-torus $T^6_{\mathrm{SU}(3)^3}$, yielding equally 27 conical singularities on the orbifold $T^6/\mathbb{Z}_3$.

9.6. Example with 2d toric singularities: $T^6/\mathbb{Z}_2$-orbifold

The reader might object that the $\mathbb{Z}_2$-orbifold does not appear in tab. 9.1, but it is very instructive to consider this particular example. Consider the flat 6-torus T^6 with the integer lattice $\Lambda = \mathbb{Z}^6$. With respect to real coordinates $X^1, \ldots, X^6$, the group's only non-trivial element $\theta \in \mathbb{Z}_2$ is defined to act like

$$\theta X^i := \begin{cases} -X^i & : \quad i = 1, \ldots, 4 \\ +X^i & : \quad i = 5, 6 \end{cases},$$

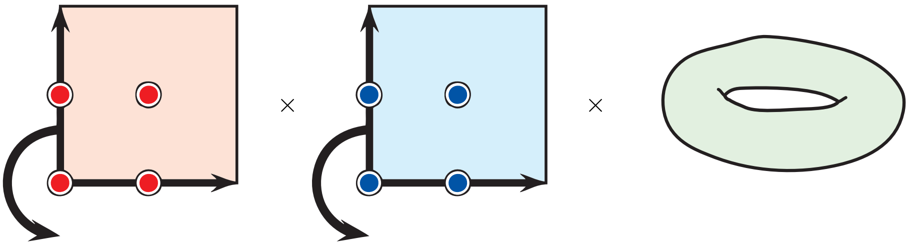


FIGURE 9.3. The $T^6/\mathbb{Z}_2$ with the specified action has 16 fixed T^2-tori, which are found in the third copy of T^2.

i.e. it has the effect of a reflection (or 180° rotation) on the first two tori and is the identity on the last one. Thus, $\theta^2 = \mathrm{Id}$, such that the group structure is provided. Due to the particular chosen lattice $\Lambda = \mathbb{Z}^6$ the fixpoint equation (9.2) in this case reduces to

$$(\mathrm{Id} - \theta)X^i_{\mathrm{fp}} = \begin{cases} 2X^i_{\mathrm{fp}} & : \quad i = 1, \ldots, 4 \\ 0 \cdot X^i_{\mathrm{fp}} & : \quad i = 5, 6 \end{cases} \in \Lambda = \mathbb{Z}^6.$$

The last two components X^5, X^6 fulfill the fixpoint condition for any value since $0 \in \Lambda$. For the first four components $X^1, \ldots, X^4$ all half-integers are admissible, such that the fixpoints can be written as

$$X_{\mathrm{fp}} = \left(\frac{\xi_1}{2}, \frac{\xi_2}{2}, \frac{\xi_3}{2}, \frac{\xi_4}{2}, X^5, X^6 \right)$$

for $\xi_1, \ldots, \xi_4 \in \mathbb{Z}$. Identifying out $\mathbb{Z}^6$ from $\mathbb{R}^6$, i.e. effectively restricting $\xi_1, \ldots, \xi_4$ to be either 0 or $\frac{1}{2}$, this yields 4 fixed points for each of the the first two tori (components $X^1, \ldots, X^4$). Since the last two components are kept invariant, they yield a **fixed torus**, such that in total there are $4 \cdot 4 = 16$ fixed tori on the $\mathbb{R}^6/(\mathbb{Z}^6 \ltimes \mathbb{Z}_2) \cong T^6/\mathbb{Z}_2$ orbifold.

9.7. Orbifold compactification of heterotic strings

In sec. 8.4 the boundary conditions of the heterotic string got modified upon toroidal compactification, such that the string remains still closed with respect to the torus lattice. As observed, this yields the additional freedom of winding states, i.e. closed strings which encircle the internal tori. The same is also happening for (toroidal) orbifolds, however, due to the point group rotations, the boundary conditions are further modified as follows:

flat space: $\begin{cases} X^i(\tau, \sigma + 2\pi) = X^i(\tau, \sigma) \\ \psi^i_{\mathrm{R}}(\tau, \sigma + 2\pi) = \pm\psi^i_{\mathrm{R}}(\tau, \sigma) \end{cases}$

compactified on torus: $\begin{cases} X^i(\tau, \sigma + 2\pi) = X^i(\tau, \sigma) + 2\pi W^i \\ \psi^i_{\mathrm{R}}(\tau, \sigma + 2\pi) = \pm\psi^i_{\mathrm{R}}(\tau, \sigma) \end{cases}$

compactified on orbifold: $\begin{cases} X^i(\tau, \sigma + 2\pi) = \left[(\ell, \theta^k) \cdot X(\tau, \sigma)\right]^i = \left[\theta^k X(\tau, \sigma)\right]^i + \ell^i \\ \psi^i_{\mathrm{R}}(\tau, \sigma + 2\pi) = \pm\left[(\ell, \theta^k)\psi_{\mathrm{R}}(\tau, \sigma)\right]^i = \pm\left[\theta^k \psi_{\mathrm{R}}(\tau, \sigma)\right]^i \end{cases}$

for $(\ell, \theta^k) \in \Lambda \ltimes P = S$. Obviously, the subgroup consisting of pure translations $(\ell, e) \in S$ reduces to the familiar case of toroidal compactification. This is called the **untwisted sector**. For non-trivial rotations $e \neq \theta \in P$, **twisted sectors** arise, which should be viewed as an additional degree of freedom—much like the winding states in toroidal compactification. The twisted sectors contain new states—twisted states—which are not present in the toroidal compactification. Essentially, the action of the point group closes strings, which would be open in the flat or toroidally compactified theory.

Due to the changed boundary conditions in the twisted sectors, the mode expansions (7.4) and (7.5) of the bosonic and fermionic fields are modified. Since the action of the point group equals a discrete rotation, any string that is closed by this action must be located around the fixpoint z_{fp} of an conical singularity. **Twisted closed strings** are thus not allowed to roam freely on the internal orbifold. The derivation of the mode expansions for the strings closed upon the action of $(\ell, \theta^k) \in S$ is quite lengthy, but should not be avoided for a full understanding. Consider the real untwisted mode expansion from chap. 7

$$X^i(\tau,\sigma) = x_0^i + p^i\tau + \sum_{n>0} \frac{1}{n}\Big[a_{\mathrm{L},n}^i \sin\big(n(\tau+\sigma)\big) + b_{\mathrm{L},n}^i \cos\big(n(\tau+\sigma)\big) \\ + a_{\mathrm{R},n}^i \sin\big(n(\tau-\sigma)\big) + b_{\mathrm{R},n}^i \cos\big(n(\tau-\sigma)\big)\Big].$$

Since the space group S acts linearly on the internal bosonic components $X(\tau,\sigma)$, one can considers the three different types of terms separately:

- *Center-of-mass*: As the string's center-of-mass x_0^i is subject to the condition

$$x_0^i = (\theta^k x_0)^i + \sum_{j=1}^{6} n_j e_j^i \qquad \text{for } k \neq 0 \text{ and } i = 1, \ldots, 6,$$

  it has to be one of the fixpoints X_{fp} of the space group action. In particular, this implies that the center-of-mass for a twisted string is fixed, such that a twisted closed string cannot move away from its respective fixpoint.
- *Momentum*: The previous statement is also reflected in the momentum p, which must satisfy

$$p^i = (\theta^k p)^i \qquad \text{for } k \neq 0 \text{ and } i = 1, \ldots, 6,$$

  which only holds for $p = 0$. Thus, a twisted closed string does not have any momentum in the orbifold components, however, it may move freely in the uncompactified 4d directions.
- *Oscillators*: For the real oscillators a_n^i and b_n^i one has to expect a certain "twisting" or "mixing" due to the P-action. A suitable ansatz is provided by a rotation

$$\underbrace{\theta^k \begin{pmatrix} a_n \\ \hline b_n \end{pmatrix} = \left(\begin{array}{c|c} \cos(2\pi n) & \sin(2\pi n) \\ \hline -\sin(2\pi n) & \cos(2\pi n) \end{array}\right) \begin{pmatrix} a_n \\ \hline b_n \end{pmatrix}}_{\text{real oscillators } \alpha_n^i \,=\, a_n^i \,-\, \mathrm{i} b_n^i} \iff \underbrace{\theta^k \alpha_n = \mathrm{e}^{2\pi n \mathrm{i}} \alpha_n}_{\text{complex oscillators}}$$

  depending on the mode excitation n. For $n \in \mathbb{Z}$ the mixing disappears, such that θ^k must be the identity. Conversely, for $\theta^k \neq e$ one expects fractional values of n. Since θ^k acts block-diagonally by rotations on the real coordinates (recall sec. 9.4), one can restrict attention to the first two components $i = 1, 2$, which describe the first 2-torus in T^6. The first "component" equation of the above oscillator mixing then reads

$$(\theta^k a_n)^i = a_n^i \cos(2\pi n) + b_n^i \sin(2\pi n)$$
$$\overset{\text{first } T^2}{\iff} \vartheta_1^k \begin{pmatrix} a_n^1 \\ a_n^2 \end{pmatrix} = \begin{pmatrix} a_n^1 \cos(2\pi\eta_1^k) - a_n^2 \sin(2\pi\eta_1^k) \\ a_n^1 \sin(2\pi\eta_1^k) + a_n^2 \cos(2\pi\eta_1^k) \end{pmatrix} = \begin{pmatrix} a_n^1 \cos(2\pi n) + b_n^1 \sin(2\pi n) \\ b_n^2 \sin(2\pi n) + a_n^2 \cos(2\pi n) \end{pmatrix}$$

  and yields the conditions $-a_n^2 = b_n^1$ and $a_n^1 = b_n^2$. Furthermore, the expected fractional value of n turns out to be

$$n_a = m + \eta_a^k \qquad \text{for } m \in \mathbb{Z} \text{ and } a = 1, 2, 3$$

  which may be different for each complex component. One arrives at the same conclusion by considering the second "mixing" equation. As mentioned, this generalizes to the other torus components $i = 1, \ldots, 6$.

This completes the derivation of the twisted string mode expansion, which is most economically written in terms of the complex coordinates

$$
\begin{aligned}
Z^a(\tau,\sigma) &= z^a_{\text{fp}} + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\left(\frac{\beta^a_{\mathrm{L},n-\eta^k_a}}{n-\eta^k_a}\mathrm{e}^{-\mathrm{i}(n-\eta^k_a)(\tau+\sigma)} + \frac{\beta^a_{\mathrm{R},n+\eta^k_a}}{n+\eta^k_a}\mathrm{e}^{-\mathrm{i}(n+\eta^k_a)(\tau-\sigma)}\right) \\
Z^{\bar a}(\tau,\sigma) &= z^{\bar a}_{\text{fp}} + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\left(\frac{\beta^{\bar a}_{\mathrm{L},n+\eta^k_a}}{n+\eta^k_a}\mathrm{e}^{-\mathrm{i}(n+\eta^k_a)(\tau+\sigma)} + \frac{\beta^{\bar a}_{\mathrm{R},n-\eta^k_a}}{n-\eta^k_a}\mathrm{e}^{-\mathrm{i}(n-\eta^k_a)(\tau-\sigma)}\right),
\end{aligned}
\tag{9.3}
$$

where the mode operators $\beta^a_{\mathrm{L},n-v_a}$, $\beta^a_{\mathrm{R},n+v_a}$, $\beta^{\bar a}_{\mathrm{L},n+v_a}$ and $\beta^{\bar a}_{\mathrm{R},n+v_a}$ correspond to the complex oscillators α^i_n from sec. 7.4 and obey the modified commutation relations

$$
\begin{aligned}
\left[\beta^a_{\mathrm{L},m-\eta^k_a}, \beta^{\bar b}_{\mathrm{L},n+\eta^k_b}\right] &= \delta^{ab}(n-\eta^k_a)\delta_{m,-n} \\
\left[\beta^a_{\mathrm{R},m+\eta^k_a}, \beta^{\bar b}_{\mathrm{R},n-\eta^k_b}\right] &= \delta^{ab}(n+\eta^k_a)\delta_{m,-n}.
\end{aligned}
$$

Obviously, the net effect is a shift of the mode excitation numbers by $\pm v_a$ for each complex component $a = 1, 2, 3$. As a side effect, the above derivation also yields the transformation behavior of the oscillators under the space group, which will be important in sec. 9.12.

For the right-moving fermionic fields, the mode expansions for the twisted sectors are derived in a similar fashion, ultimately leading to

$$
\underbrace{\begin{aligned}
\psi^a_{\mathrm{R}}(\tau,\sigma) &= \sum_{n\in\mathbb{Z}} e^a_{n+\eta^k_a}\mathrm{e}^{-\mathrm{i}(n+\eta^k_a)(\tau-\sigma)} \\
\psi^{\bar a}_{\mathrm{R}}(\tau,\sigma) &= \sum_{n\in\mathbb{Z}} e^{\bar a}_{n-\eta^k_a}\mathrm{e}^{-\mathrm{i}(n-\eta^k_a)(\tau-\sigma)}
\end{aligned}}_{\text{Ramond (R)}}
\quad\text{and}\quad
\underbrace{\begin{aligned}
\psi^a_{\mathrm{R}}(\tau,\sigma) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} c^a_{r+\eta^k_a}\mathrm{e}^{-\mathrm{i}(r+\eta^k_a)(\tau-\sigma)} \\
\psi^{\bar a}_{\mathrm{R}}(\tau,\sigma) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} c^{\bar a}_{r-\eta^k_a}\mathrm{e}^{-\mathrm{i}(r-\eta^k_a)(\tau-\sigma)},
\end{aligned}}_{\text{Neveu-Schwarz (NS)}}
$$

where a shifting of the modes is again observed. However, in contrast to the bosonic fields, the shift in the mode expansion operators does not affect the anti-commutation relations

$$
\begin{aligned}
\left\{e^a_{m+\eta^k_a}, e^{\bar b}_{n-\eta^k_b}\right\} &= \delta^{ab}\delta_{m,-n} \\
\left\{c^a_{r+\eta^k_a}, c^{\bar b}_{s-\eta^k_b}\right\} &= \delta^{ab}\delta_{r,-s}
\end{aligned}
$$

of the fermionic oscillators, i.e. the excitation numbers are unchanged. One can derive similar mode expansions for the strings closed by the actions of elements (θ^k, θ^l) in the case of double-cyclic point groups $\mathbb{Z}_m \times \mathbb{Z}_n$.

9.8. Embedding of the space group into the gauge degrees of freedom

Due to the action of the point group and the drastic change in the internal space-time geometry, one has to address the issue of modular invariance again (recall sec. 7.5), i.e. the consistency of one-loop string diagrams. This is done by embedding the orbifold space group S into the gauge degrees of freedom, such that certain consistency conditions are satisfied, which will be discussed in the next section.

More precisely, it is only required to embed the point subgroup $P \subset S$, however, one may also embed the lattice vectors $\Lambda \subset S$—viewed as an additive subgroup—as non-trivial background fields. From a certain perspective, the general concept seems to be reminiscent of the "embedding of the spin connection" discussed in the context of Calabi-Yau compactifications in sec. 8.7, however, there are profound technical differences. Nevertheless, for each case the net effect is an embedding of the respective structure (the point group or the "spin connection") that carries the information on the holonomy of the internal space.

Let $\mathbb{Z}_N = \{\theta^k\}$ and $\mathbb{Z}_N \times \mathbb{Z}_M = \{(\theta^k_1, \theta^l_2)\}$ be the respective point groups, such that any element can be simply written as θ^k or (θ^k, θ^l). Furthermore, let $V \in \Lambda_{\mathrm{E}_8\times\mathrm{E}_8}$ or $V_1, V_2 \in \Lambda_{\mathrm{E}_8\times\mathrm{E}_8}$ denote (linearly independent) shift vectors on the $\mathrm{E}_8\times\mathrm{E}_8$-torus lattice. Let the

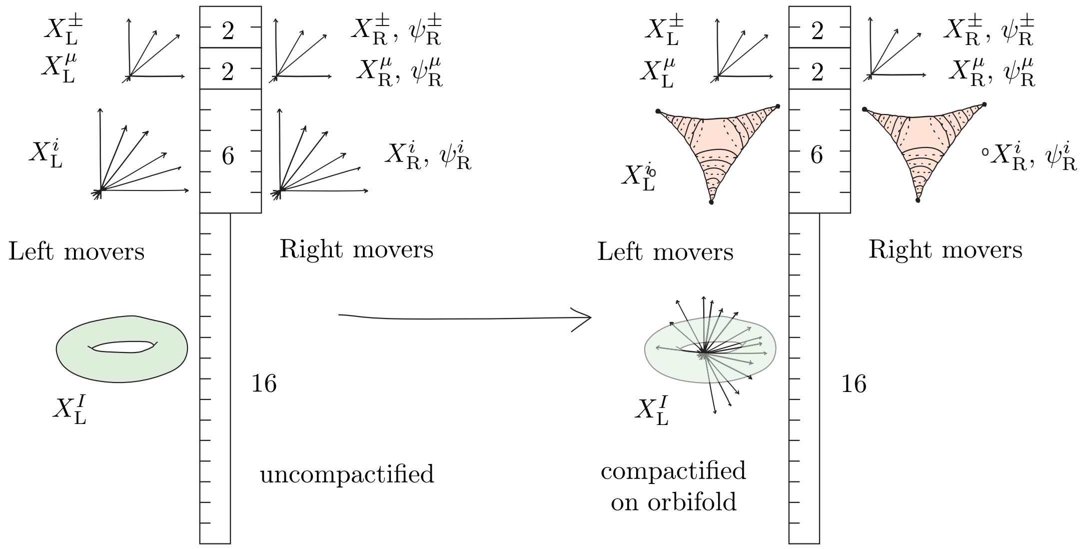


FIGURE 9.4. Orbifold compactification of the heterotic string.

orbifold's underlying torus lattice $\Lambda = \Lambda_6$ be generated by the lattice basis vectors e_i for $i = 1, \ldots, 6$, such that any lattice vector $\ell \in \Lambda$ may be written as

$$\ell = \sum_{i=1}^{6} n_i e_i \qquad \text{for } n_i \in \mathbb{Z}.$$

The possible additional 16-component constant background fields A_i, $i = 1, \ldots, 6$, for the (optional) embedding of the torus lattice vectors $\ell \in \Lambda$ are called **Wilson lines**. This was first considered in [INQ87a] in the context of gauge symmetry breaking. Furthermore, Wilson lines allow to control the number of matter multiplets, which will be important in the semi-realistic $\mathbb{Z}_6$-II orbifold model considered in the next chapter. The embeddings of the space group are homomorphisms $\Xi : S = \Lambda \ltimes P \longrightarrow G$, which can be made explicit as follows:

cyclic group $\mathbb{Z}_N$: $$\left(\sum_{i=1}^{6} n_i e_i, \theta^k\right) \mapsto \Xi(\ell, \theta^k) := \left(\sum_{i=1}^{6} n_i A_i, kV\right)$$

double-cyclic g. $\mathbb{Z}_N \times \mathbb{Z}_M$: $$\left(\sum_{i=1}^{6} n_i e_i, (\theta_1^k, \theta_2^l)\right) \mapsto \Xi\big(\ell, (\theta_1^k, \theta_2^l)\big) := \left(\sum_{i=1}^{6} n_i A_i, kV_1 + lV_2\right).$$

The sole embedding of the point group P without additional Wilson lines is usually done via the **standard embedding**. Using the $\mathbb{Z}_N$-generating vectors v and w from tab. 9.1, define a 16-component shift vector

$$V_v := (v_1, v_2, v_3, 0^{13}) = \overbrace{(v_1, v_2, v_3, 0, 0, 0, 0, 0)}^{\text{first } \mathrm{E}_8\text{-copy}}\overbrace{(0, 0, 0, 0, 0, 0, 0, 0)}^{\text{second } \mathrm{E}_8\text{-copy}},$$

which is then used as either $V = V_v$ or $V_1 = V_v$, $V_2 = V_w$ in the cases above. Given an element $g = \big(\sum_{i=1}^{6} n_i A_i, kV\big) = \Xi(\ell, \theta^k) \in S$, its action on the gauge degrees of freedom is[c]

$$gX_{\mathrm{L}}^{I} = X_{\mathrm{L}}^{I} + 2\pi \left(kV^I + \sum_{i=1}^{6} n_i A_i^I\right). \tag{9.4}$$

[c]Obviously from (9.4), the group G has a simple additive action on the gauge degrees of freedom X_{L}^{I}. In essence, the embedding of S into the group G is nothing else than a fancy way to define an action of S on the gauge degrees of freedom. However, this particular formulation emphasizes the "embedding of the holonomy-carrying structure", as mentioned before.

The additive group G is often called the **gauge twisting group** and its elements are the **gauge twists**. The twisted boundary conditions of the internal gauge degrees of freedom are

$$\begin{aligned} X_{\mathrm{L}}^I(\tau,\sigma+2\pi) &\equiv \left[(\ell,\theta^k)\cdot X_{\mathrm{L}}(\tau,\sigma+2\pi)\right]^I \bmod 2\pi\Lambda_{\mathrm{E}_8\times\mathrm{E}_8} \\ &\equiv X_{\mathrm{L}}^I(\tau,\sigma)+2\pi\left(kV^I+\sum_{i=1}^{6} n_i A_i^I\right) \bmod 2\pi\Lambda_{\mathrm{E}_8\times\mathrm{E}_8}, \end{aligned}$$

where the identification of $2\pi\Lambda_{\mathrm{E}_8\times\mathrm{E}_8}$ takes care of the left-/right-asymmetry arising from the unmatched left-moving momenta. In the corresponding mode expansion

$$X_{\mathrm{L}}^I(\tau,\sigma)=x_{\mathrm{L},0}^I+\left(p_{\mathrm{L}}^I+kV^I+\sum_{i=1}^{6} n_i A_i^I\right)(\tau+\sigma)+\frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{\alpha_{\mathrm{L},n}^I}{n}\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)}$$

it becomes obvious, that the embedding of the space group effects a shifting of the internal quantum numbers (or internal momenta, respectively). It is useful to define a **local gauge shift**

$$V_{k\text{-}n}:=kV+\sum_{i=1}^{6} n_i A_i$$

in this context, such that $V_{k\text{-}n}$ represents the shifting for the respective space group element. As mentioned, the presence of Wilson lines refers to the embedding of the entire space group (in a non-trivial manner) instead of the point subgroup.

9.9. Modular invariance and anomalies

While the general structure of an orbifold compactification is rather simple, the aforementioned embeddings have to satisfy certain constraints to guarantee modular invariance and to provide a well-defined anomaly-free closed string orbifold model. The discussion follows the detailed derivation found in [BL99, §2.4].

In essence, modular invariance provides the mathematical basis for a well-defined calculation of one-loop vacuum transition amplitudes, which are provided by genus-1 Riemannian surfaces (i.e. the torus T^2). Naturally, one might wonder how this generalizes to higher loop orders. This is closely related to the existence of global anomalies, which would spoil the global diffeomorphism invariance of the string worldsheet. Applying the same general techniques as Witten did for the general heterotic string (cf. sec. 7.10), one has to investigate the G-equivariant cohomology and geometry (in terms of G-equivariant characteristic classes) of the determinant line bundle. In [FV87] it was shown, that the global anomalies of all loop orders (i.e. string worldsheets of any genus) vanish if the one-loop modular invariance is satisfied.

To spare the reader the quite lengthy derivations—which are found in the mentioned literature—only the resulting constraints for $\mathbb{Z}_N$-orbifolds will be stated:

- *Well-definiteness:* Let $\mathbb{Z}_N$ be the cyclic point group of the considered toroidal orbifold $T^6/\mathbb{Z}_N$. Since the identity $\theta^N=1$ holds for any element $\theta\in P$ and the embedding into the gauge degrees of freedom is a homomorphism, for any space group element $(\ell,\theta^k)=\left(\sum_{i=1}^6 n_i e_i,\theta^k\right)\in S$ with corresponding twist $\Xi(\ell,\theta^k)=\left(\sum_{i=1}^6 n_i A_i,kV\right)$ the identities

  $$\begin{aligned} &\left(\textstyle\sum_{i=1}^6 n_i e_i,\theta^k\right)^N X^i\equiv X^i \bmod \Lambda_6\text{-shifts} && i=1,\dots,6 \\ &\left(\textstyle\sum_{i=1}^6 n_i A_i,kV\right)^N X_{\mathrm{L}}^I\equiv X_{\mathrm{L}}^I \bmod \Lambda_{\mathrm{E}_8\times\mathrm{E}_8}\text{-shifts} && I=1,\dots,16 \end{aligned}$$

  must be satisfied for $k=0,\dots,N-1$. Using the explicit action (9.4) of the embedded space group (i.e. the gauge twisting group), the Wilson lines A_i and the embedding shift vector V are constrained to satisfy $NV_{k\text{-}n} \in \Lambda_{\mathrm{E}_8 \times \mathrm{E}_8}$, which implies the constraints

(9.5)
$$\begin{aligned} NV &\in \Lambda_{\mathrm{E}_8 \times \mathrm{E}_8} \\ NA_i &\in \Lambda_{\mathrm{E}_8 \times \mathrm{E}_8} \qquad \text{for } i = 1, \dots, 6. \end{aligned}$$

In general, the rotational action of a non-prime point group $P = \mathbb{Z}_N$ on the three planes of the 6-torus may reduce to a $\mathbb{Z}_n$-rotation for $n < N$ on some T^2-planes, such that $mn = N$ for some $m \in \mathbb{N}$. It is useful to introduce the notion of a **Wilson line of order n**, which obeys the stronger condition

$$nA_i \in \Lambda_{\mathrm{E}_8 \times \mathrm{E}_8} \qquad \text{for } i = 1, \dots, 6.$$

- *Modular invariance:* A one-loop vacuum-vacuum transition amplitude (i.e. creation and annihilation of a particle/antiparticle pair) is described by a torus worldsheet in string theory. The corresponding worldsheet coordinates (τ, σ) are only defined up to modular transformations, which constitute the (discrete) matrix group $\mathrm{SL}(2;\mathbb{Z})$, see [BBS07, §3.5] or [BL99, §2.3]. In the perturbative theory, the partition function provides the means to calculate amplitudes. To ensure its invariance under modular transformations, the gauge embedding has to satisfy

$$N\Big[(V_{k\text{-}n})^2 - (kv)^2\Big] \equiv 0 \bmod 2 \qquad \text{for } k = 0, \dots, N-1,$$

where $v = (v_1, v_2, v_3)$ is the $\mathbb{Z}_N$-generating vector from tab. 9.1, see [BL99, §2.4] for details. This condition is equivalent to the simpler constraints

$$\begin{aligned} N(V^2 - v^2) &\equiv 0 \bmod 2 \\ NA_i^2 &\equiv 0 \bmod 2 \qquad \text{for } i = 1, \dots, 6 \\ N\langle V, A_i\rangle_{\mathrm{L}} &\equiv 0 \bmod 1 \qquad \text{for } i = 1, \dots, 6 \\ N\langle A_i, A_j\rangle_{\mathrm{L}} &\equiv 0 \bmod 1 \qquad \text{for } i \neq j = 1, \dots, 6, \end{aligned}$$

which are usually referred to as the **weak modular invariance conditions**, where $\langle p, \rho\rangle_{\mathrm{L}} := \sum_{I=1}^{16} p_i \rho_i$ is the standard inner product on the momentum vectors of the gauge degrees of freedom. As stated before, those constraints also guarantee anomaly-freedom of the orbifold compactified theory for arbitrary worldsheets, see [FV87]. By adding suitable $\mathrm{E}_8 \times \mathrm{E}_8$ root lattice vectors to the embedding vector V and the Wilson lines A_i, it can be shown that those obey

(9.6)
$$\begin{aligned} V^2 - v^2 &\equiv 0 \bmod 2 \\ A_i^2 &\equiv 0 \bmod 2 \qquad \text{for } i = 1, \dots, 6 \\ \langle V, A_i\rangle_{\mathrm{L}} &\equiv 0 \bmod 1 \qquad \text{for } i = 1, \dots, 6 \\ \langle A_i, A_j\rangle_{\mathrm{L}} &\equiv 0 \bmod 1 \qquad \text{for } i \neq j = 1, \dots, 6, \end{aligned}$$

which are called the **strong modular invariance conditions**. Returning to the original equation, those separate conditions can be written as

$$(V_{k\text{-}n})^2 - (kv)^2 \equiv 0 \bmod 2 \qquad \text{for } k = 0, 1.$$

The separate conditions (9.6) allow for a relatively easy analysis of the physical states surviving the corresponding consistency projection.

To summarize: In order to fully specify an **toroidal orbifold compactification** of the heterotic string, one has to supply a cyclic or double-cyclic point group P from tab. 9.1 and a suitable 6-dimensional lattice $\Lambda = \Lambda_6$, such that $O = \mathbb{R}^6/S = T^6_\Lambda/P$ defines the (singular) geometry of the internal space. One also has to specify an additive action of the point group (or the entire space group $S = \Lambda \ltimes P$) on the gauge degrees of freedom, which is done by providing an "embedding" homomorphism $\xi := \Xi|_P : P \longrightarrow G$ (or $\Xi : S \longrightarrow G$) subject to the consistency conditions (9.5) an (9.6).

9.10. Action of the space group on massless states

The heterotic particle spectrum arises from applying the oscillators on the ground state. More precisely, there is a left-moving and right-moving ground state on which the respective oscillators are independently applied. The linking object is the level matching condition, which requires the left- and right-moving mass to be equal. This general concept still holds for the orbifold compactified theory, however, the states have to be invariant under the action of the space group. Furthermore, the ground states are dependent on the respective particle sector, i.e. the Hilbert space of untwisted strings has a different ground state compared to the Hilbert spaces of the twisted sectors. This will be investigated in more detail in the following two sections.

The essential effect of applying the space group on a state is the appearance of a non-trivial complex phase. Let $|p\rangle_\mathrm{L}$ denote the left-moving ground state whose internal quantum numbers $p = p^I$ for $I = 1, \dots, 16$ stem from the quantized momenta of the gauge degrees of freedom. The right-moving ground state $|q\rangle_\mathrm{R}$ consists of an (half-)integer weight vector which represents a Ramond or Neveu-Schwarz ground state. The complete ground state is then a tensor product of the form

$$|p\rangle_\mathrm{L} \otimes |q\rangle_\mathrm{R},$$

however, for general values of $p = p^I$ this does not satisfy the level matching condition. This was discussed at length in sec. 7.7.

Given an space group element $(\ell, \theta^k) = \left(\sum_{i=1}^6 n_i e_i, \theta^k\right) \in S$ with the corresponding gauge twisting $\Xi(\ell, \theta^k) = \left(\sum_{i=1}^6 n_i A_i, kV\right) \in G$, it follows the transformation behavior

$$(\ell, \theta^k) : \begin{cases} |p\rangle_\mathrm{L} \mapsto \mathrm{e}^{2\pi\mathrm{i}\langle p, V_{k\text{-}n}\rangle_\mathrm{L}} |p\rangle_\mathrm{L} \\ |q\rangle_\mathrm{R} \mapsto \mathrm{e}^{2\pi\mathrm{i}\langle q, kv\rangle_\mathrm{R}} |q\rangle_\mathrm{R}, \end{cases}$$

where $\langle q, \varsigma\rangle_\mathrm{R} := \sum_{a=0}^3 \bar{q}_a \varsigma_a$ is the standard inner product on the weight lattice describing the right-moving ground state. Note that the 3-component $\mathbb{Z}_N$-generating vector $v = (v_1, v_2, v_3)$ is extended by a fourth component, such that for $v = (0; v_1, v_2, v_3)$ one can take the inner product with a state-defining vector q from sec. 7.6.

Thus, the net effect of an element of the space group on a heterotic particle ground state is an additional phase

$$\mathrm{e}^{2\pi\mathrm{i}[\langle p, V_{k\text{-}n}\rangle_\mathrm{L} + \langle q, kv\rangle_\mathrm{R}]}.$$

However, in general the oscillators will not be invariant under the action of the space group, which yields additional phases for the excited twisted states. Those states invariant under the action of the space group (i.e. states with a trivial phase) are well-defined on the underlying orbifold space and thus are still present in the orbifold compactified theory. States not invariant under the S-action have to be projected out, as they are not consistently defined on the orbifold geometry. Depending on the point group twist and the respective fixpoint, the boundary conditions of the closed string are affected differently. For the neutral element $e \in P$ they simply reduce to the toroidal boundary conditions, whereas any other element closes strings, which would be open in the toroidally compactified theory. Thus, the massless particle spectrum of the heterotic string compactified on an orbifold can be separated in two categories discussed in the following sections.

9.11. Untwisted sector and breaking of the effective 4d SUSY

The **untwisted massless sector** consists of those states which are already closed in the uncompactified theory, however, as seen in the last section certain additional constraints have to be satisfied. Recall from tab. 7.2 on p. 84 that the massless heterotic particle spectrum consists of states

$$|p\rangle_\mathrm{L} \otimes |q\rangle_\mathrm{R} \qquad \text{and} \qquad \alpha_{\mathrm{L},-1}|0\rangle_\mathrm{L} \otimes |q\rangle_\mathrm{R} \quad \begin{cases} \alpha^\mu_{\mathrm{L},-1}|0\rangle_\mathrm{L} \otimes |q\rangle_\mathrm{R} & \text{for } \mu = 1, \dots, 8 \\ \alpha^I_{\mathrm{L},-1}|0\rangle_\mathrm{L} \otimes |q\rangle_\mathrm{R} & \text{for } I = 1, \dots, 16, \end{cases}$$

where $p_{\mathrm{L}}^2 = p_{\mathrm{L}}^I p_{\mathrm{L}}^I = 2$ holds. In the orbifold compactified theory one only keeps the P-invariant massless states of the toroidally compactified heterotic string, or equivalently the S-invariant states of the flat theory. Note that this excludes winding states from the toroidally compactified theory, since those are massive.

The **supergravity multiplet** consists exactly of the states $\alpha^\mu_{\mathrm{L},-1}|0\rangle_{\mathrm{L}} \otimes |q\rangle_{\mathrm{R}}$ for components $\mu = 1, \ldots, 8$, as seen from tab. 7.2 on p. 84. According to the previous section, those states transform like

$$\begin{aligned}\alpha^\mu_{\mathrm{L},-1}|0\rangle_{\mathrm{L}} \otimes |q\rangle_{\mathrm{R}} &\mapsto \mathrm{e}^{2\pi\mathrm{i}[\langle 0, V_{k\text{-}n}\rangle_{\mathrm{L}} + \langle q, kv\rangle_{\mathrm{R}}]}\alpha^\mu_{\mathrm{L},-1}|0\rangle_{\mathrm{L}} \otimes |q\rangle_{\mathrm{R}} \\ &= \mathrm{e}^{2\pi\mathrm{i}k\langle q,v\rangle_{\mathrm{R}}}\alpha^\mu_{\mathrm{L},-1}|0\rangle_{\mathrm{L}} \otimes |q\rangle_{\mathrm{R}}\end{aligned}$$

under the action of the space group. To remove the complex phase, the condition $\langle q, v\rangle_{\mathrm{R}} \in \mathbb{Z}$ has to be satisfied, where q takes the discrete momenta (7.7) and $v = (0; v_1, v_2, v_3)$ is the point group generating vector. From the original 16 possible vectors q only the four possibilities

$$\begin{aligned}\text{Neveu-Schwarz:} \quad q &= \Big(+1; 0, 0, 0\Big) \quad \text{or} \quad \Big(-1; 0, 0, 0\Big) \\ \text{Ramond:} \quad q &= \Big(+\tfrac{1}{2}; +\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2}\Big) \quad \text{or} \quad \Big(-\tfrac{1}{2}; -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\Big)\end{aligned} \tag{9.7}$$

remain. Thus, only two bosonic (Neveu-Schwarz) and fermionic (Ramond) ground states remain, such that from the original 64 bosonic/fermionic degrees of freedom in $\alpha^\mu_{\mathrm{L},-1}|0\rangle_{\mathrm{L}} \otimes |q\rangle_{\mathrm{R}}$ only 16 remain after the orbifold projection. In particular, the two fermionic ground states actually correspond to the two different 4d chiralities, see sec. 7.8. After applying the Kaluza-Klein mechanism, the 16 bosonic and fermionic degrees of freedom give rise to the 4d $\mathcal{N} = 1$ SUGRA multiplet, one 4d $\mathcal{N} = 1$ chiral multiplet containing the dilaton and six 4d $\mathcal{N} = 1$ SYM multiplets, all of which are listed in sec. 6.5.

It is important to reflect about the breaking of the supersymmetry in the orbifold case: In the toroidal compactification (carried out in sec. 8.4) four copies of the gravitino remained, since none of the supersymmetry was broken. The additional requirement of S-invariance then projects out three of those four gravitini—leaving only $\mathcal{N} = 1$ instead of $\mathcal{N} = 4$ 4d SUSY. However, this is in fact due to the SU(3)-holonomy of the considered Calabi-Yau orbifolds, which is represented in the condition $v_1 + v_2 + v_3 = 0$ imposed in sec. 9.4 on the $\mathbb{Z}_N$-generating vector v. The breaking of the supersymmetry can thus be attributed to the Calabi-Yau condition, again, albeit it reveals itself in a very different manner compared to the manifold case considered in chap. 8.

The next set of massless heterotic states are the 16 **uncharged $\mathbf{E_8} \times \mathbf{E_8}$-super-Yang-Mills multiplets** contained in the states $\alpha^I_{\mathrm{L},-1}|0\rangle_{\mathrm{L}} \otimes |q\rangle_{\mathrm{R}}$, which transform as

$$\begin{aligned}\alpha^I_{\mathrm{L},-1}|0\rangle_{\mathrm{L}} \otimes |q\rangle_{\mathrm{R}} &\mapsto \mathrm{e}^{2\pi\mathrm{i}[\langle 0, V_{k\text{-}n}\rangle_{\mathrm{L}} + \langle q, kv\rangle_{\mathrm{R}}]}\alpha^I_{\mathrm{L},-1}|0\rangle_{\mathrm{L}} \otimes |q\rangle_{\mathrm{R}} \\ &= \mathrm{e}^{2\pi\mathrm{i}k\langle q,v\rangle_{\mathrm{R}}}\alpha^I_{\mathrm{L},-1}|0\rangle_{\mathrm{L}} \otimes |q\rangle_{\mathrm{R}}.\end{aligned}$$

This gives the same conditions as for the SUGRA multiplet, such that (9.7) also lists the possible right-moving ground states in this case, yielding 4d $\mathcal{N} = 1$ gauge bosons and gaugini in the effective 4d theory. However, in this approach the gauge group is not reduced. One has to embed the space group $S = \Lambda \ltimes P$ in a certain way into the gauge degrees of freedom to achieve an actual reduction of the rank of the gauge algebra, see [INQ87b] for further details.

The 480 **charged gauge bosons** that arise from non-trivial momenta of the gauge degrees of freedom transform as

$$|p^I\rangle_{\mathrm{L}} \otimes |q\rangle_{\mathrm{R}} \mapsto \mathrm{e}^{2\pi\mathrm{i}[\langle p, V_{k\text{-}n}\rangle_{\mathrm{L}} + \langle q, kv\rangle_{\mathrm{R}}]}|p^I\rangle_{\mathrm{L}} \otimes |q\rangle_{\mathrm{R}}.$$

In order to fulfill $\langle q, v\rangle \in \mathbb{Z}$, the right-moving ground state $|q\rangle_{\mathrm{R}}$ is again restricted to one of the cases (9.7). The second condition $\langle p, V_{k\text{-}n}\rangle_{\mathrm{L}} = \langle p, kV + \sum_{i=1}^6 n_i A_i\rangle_{\mathrm{L}} \in \mathbb{Z}$ must hold for all

heterotic 10d supergravity		$\Longrightarrow$	orbifold-comp. 4d theory	
1	10d $\mathcal{N}=1$ SUGRA multiplet		1	4d $\mathcal{N}=1$ SUGRA multiplet
16	10d $\mathcal{N}=1$ SYM multiplets		1	4d $\mathcal{N}=1$ chiral multiplet (dilaton)
			22	4d $\mathcal{N}=1$ SYM multiplets

TABLE 9.2. Reduction of the massless heterotic particle spectrum in the untwisted sector upon orbifold compactification.

possible values of k and n_i, which can be satisfied by the two simpler conditions

$$\langle p, V_{k\text{-}n}\rangle_{\mathrm{L}} \in \mathbb{Z} \iff \begin{array}{c}\langle p, V\rangle_{\mathrm{L}} \in \mathbb{Z} \\ \langle p, A_i\rangle_{\mathrm{L}} \in \mathbb{Z}\end{array} \qquad \text{for } i=1,\dots,6. \tag{9.8}$$

One should appreciate how the presence of Wilson lines (i.e. the non-trivial embedding of the full space group instead of its point subgroup) affects the massless particle spectrum of the untwisted sector: Due to the extra conditions (9.8), the gauge group of the charged gauge bosons is broken—however, it is a rather complicated task to determine the remaining gauge group in practice.

9.12. Twisted sectors and S-invariant states

Due to the action of the point group, the particular geometry of an orbifold allows strings to close in a way similar to the winding states in toroidal compactification. More precisely, a string may be closed by wrapping itself around one of the singularities of the internal orbifold. Let $\mathfrak{T}^k$ denote the set of ***k*-twisted strings**, i.e. strings which are closed under the action of the element $\theta^k \in P$. Furthermore, let $\mathfrak{T}^k_z$ denote those k-twisted strings which are wrapped around the singularity associated with the fixpoint[d] $z = z_{\mathrm{fp}}$, satisfying

$$z_{\mathrm{fp}} \equiv \theta^k z_{\mathrm{fp}} \bmod \Lambda_6 \iff z^a_{\mathrm{fp}} = (\theta^k z_{\mathrm{fp}})^a + \sum_{i=1}^{6} n_i e^a_i \qquad \text{for } a=1,2,3. \tag{9.9}$$

The untwisted sector discussed in the previous section corresponds to $\mathfrak{T}^0$ as its strings are already closed (i.e. closed under the trivial action of the unit element). The twisted sectors are usually numbered by the twist order k, i.e. the **first twisted sector** refers to $\mathfrak{T}^1$, however, there may be more than one fixpoint under the action of $\theta = \theta^1 \in P$, e.g. the $T^6/\mathbb{Z}_3$-orbifold investigated in sec. 9.5.

Let z_{fp} be a fixpoint under the action of $\theta^{k_{\mathrm{fp}}}$ with its unique lattice shift vector $n_{\mathrm{fp}} = (n_1,\dots,n_6) \in \mathbb{Z}^6$. In (9.3) the complex mode expansion for the twisted sectors revealed a certain left-/right-asymmetrical shift in the mode expansion. From the right-moving part of (9.3) it becomes obvious, that for a twist $\theta^{k_{\mathrm{fp}}} \in P$ there is a shift $q^a \to q^a + k_{\mathrm{fp}} v^a$. Likewise, from (9.4) it follows the shifting $p^I \to p^I + k_{\mathrm{fp}} V^I + \sum_{i=1}^6 n_i^{\mathrm{fp}} A_i^I = p^I + V^{\mathrm{fp},I}_{k\text{-}n}$ of the internal quantum numbers, both of which can be captured by introducing for each fixpoint the fixpoint's

[d]Since z_{fp} actually is a point of the torus $T^6_\Lambda = \mathbb{R}^6/\Lambda$, in flat space the point $\tilde{z}_{\mathrm{fp}} := z_{\mathrm{fp}} + \ell \in \mathbb{R}^6$ refers to the same fixpoint for any $\ell \in \Lambda$. However, in general $\tilde{z}_{\mathrm{fp}}$ has a different lattice shift vector $\tilde{n} = (\tilde{n}_1,\dots,\tilde{n}_6)$ in the explicit fixpoint formula (9.9) as z_{fp}. One can deal with this problem by defining the lattice's **fundamental region**

$$\mathcal{F}_\Lambda := \left\{ \sum_{i=1}^{6} \lambda_i e_i : \lambda_i \in [0,1[\right\} = e_1[0,1[+ \cdots + e_6[0,1[$$

spanned by the lattice basis vectors e_i, which distinguishes one of the lattice cells. If one requires $z_{\mathrm{fp}} \in \mathcal{F}_\Lambda$, the ambiguity is resolved as only a unique pair (k,n) satisfies the explicit fixpoint equation (9.9).

respective **shifted "momenta"** and **ground states**[e]

$$q_{\mathrm{fp}} := q + k_{\mathrm{fp}} \qquad \text{and states} \qquad |q_{\mathrm{fp}}\rangle_{\mathrm{R}} := |q + k_{\mathrm{fp}}\rangle_{\mathrm{R}}$$
$$p_{\mathrm{fp}} := p + V^{\mathrm{fp}}_{k\text{-}n} \qquad\qquad |p_{\mathrm{fp}}\rangle_{\mathrm{L}} := \big|p - V^{\mathrm{fp}}_{k\text{-}n}\big\rangle_{\mathrm{L}},$$

such that $|p_{\mathrm{fp}}\rangle_{\mathrm{L}} \otimes |q_{\mathrm{fp}}\rangle_{\mathrm{R}}$ is the complete ground state for the respective twisted sector $\mathfrak{T}^{k_{\mathrm{fp}}}_{z_{\mathrm{fp}}}$. It is important to note that the twist order k_{fp} and lattice shift vector n_{fp} are unique for each fixpoint z_{fp} and thus for each twisted sector. The modified (massless) mass formulas for the left- and right-moving part of a twisted closed string contained in $\mathfrak{T}^{k_{\mathrm{fp}}}_{z_{\mathrm{fp}}}$ are then given by

$$\begin{aligned} &\text{left-moving mass:} && \frac{m_{\mathrm{L}}^2}{4} = \frac{1}{2}(p_{\mathrm{L,fp}})^2 + N_{\mathrm{L}} - 1 + \delta c_{k_{\mathrm{fp}}} \\ &\text{right-moving mass:} && \frac{m_{\mathrm{R}}^2}{4} = \frac{1}{2}(q_{\mathrm{R,fp}})^2 - \frac{1}{2} + \delta c_{k_{\mathrm{fp}}} \end{aligned} \tag{9.10}$$

where δc_k represents the change of the **ground state energy**, which can be calculated to be

$$\delta c_k = \frac{1}{2}\sum_{a=0}^{3} \eta_k^a (1 - \eta_k^a).$$

Thus, the zero point energy in orbifold compactifications also depends on the considered twisted sector, more precisely, on the twist order k. The excitation number N_{L} for the left-moving twisted bosonic string takes the form

$$N_{\mathrm{L}} := \underbrace{\sum_{\mu=0}^{3}\sum_{\substack{n>0\\ n\in\mathbb{Z}}} \alpha^{\mu}_{-n}\alpha^{\mu}_{n}}_{\text{flat 4d coordinates}} + \underbrace{\sum_{a=1}^{3}\sum_{\substack{n+\eta^k_a>0\\ n\in\mathbb{Z}}} \beta^{\bar a}_{-n-\eta^k_a}\beta^{a}_{n+\eta^k_a} + \sum_{a=1}^{3}\sum_{\substack{n-\eta^k_a>0\\ n\in\mathbb{Z}}} \beta^{a}_{-n+\eta^k_a}\beta^{\bar a}_{n-\eta^k_a}}_{\text{orbifold compactified internal 6d coordinates}},$$

where $k = k_{\mathrm{fp}}$. The general excitation number operators for the right-moving part may be found in [BL99, §1.4]. As in the case of the untwisted sector, the full embedding of the space group (i.e. the presence of non-trivial Wilson lines A_i) directly affects the corresponding physics, in this case the mass formula of the left-moving part of the heterotic string. One should compare the results (9.10) with the uncompactified case (7.6).

Finally, the effect of the space group on the twisted states can be made explicit in the same fashion as before, i.e. given an space group element $(\tilde{\ell}, \theta^{\tilde{k}}) \in S$, there is a complex phase

$$|p_{\mathrm{fp}}\rangle_{\mathrm{L}} \mapsto \mathrm{e}^{2\pi\mathrm{i}\langle p_{\mathrm{fp}}, \tilde{V}_{k\text{-}n}\rangle_{\mathrm{L}}} |p_{\mathrm{fp}}\rangle_{\mathrm{L}}$$
$$|q_{\mathrm{fp}}\rangle_{\mathrm{R}} \mapsto \mathrm{e}^{2\pi\mathrm{i}\langle q_{\mathrm{fp}}, \tilde{k}v\rangle_{\mathrm{R}}} |q_{\mathrm{fp}}\rangle_{\mathrm{R}}$$

with respect to the shifted momenta and ground states. The remaining states of the orbifold compactification are again those states invariant under the action of the space group S, i.e. the states with a trivial complex phase. However, as it was already observed in the derivation of the twisted string mode expansion in sec. 9.7, the twisted oscillators show a non-trivial transformation behavior

$$\begin{aligned} \beta^{a}_{n-\eta^k_a} &\mapsto \mathrm{e}^{2\pi\mathrm{i}\eta^{\tilde{k}}_a}\beta^{a}_{n-\eta^k_a} \\ \beta^{\bar a}_{n-\eta^k_a} &\mapsto \mathrm{e}^{-2\pi\mathrm{i}\eta^{\tilde{k}}_a}\beta^{\bar a}_{n-\eta^k_a} \end{aligned} \qquad \text{for } \eta^{\tilde{k}}_a := \tilde{k}v_a \bmod 1,$$

which reflects an transformation behavior of either the fundamental or conjugate representation of SU(3).

[e]The negative sign comes from $|p_{\mathrm{fp}}\rangle_{\mathrm{L}}$ being a left-moving state, whereas $|q_{\mathrm{fp}}\rangle_{\mathrm{R}}$ is right-moving. In comparison with the mode expansions carried out before, the R-moving part always has a positive sign, whereas the L-moving part has a negative sign. The shifted "momenta" p_{fp} and q_{fp} are both introduced as being R-moving, such that the sign comes from

$$p_{\mathrm{L,fp}} = -p_{\mathrm{R,fp}} = -\left(p_{\mathrm{R}} + V_{\mathrm{fp}}\right) = -p_{\mathrm{R}} - V_{\mathrm{fp}} = p_{\mathrm{L}} - V_{\mathrm{fp}}.$$

For actual computations, due to the shifts in the momenta, the resulting projection conditions are much harder to solve compared to the case of the untwisted sector. However, using the strong modular invariance conditions (9.6), it can be shown that all massless states in the first twisted sector are actually S-invariant. For prime orbifolds $T^6/\mathbb{Z}_N$, i.e. where the order N on the point group is a prime integer, this statement generalizes to all twisted sectors. For non-prime orbifolds one has to investigate certain sub-twists in order to satisfy this condition, see [BHLR06b, §2.5.2] where it is shown how this can be carried out in principle.

9.13. Hilbert spaces of twisted string states

The particular form of the Hilbert spaces of states remains to be investigated, following the original treatment in [DHVW86, §3]. For each space group element $g = (\ell, \theta^k) \in S$ let $\mathcal{H}_g$ denote the Hilbert space of strings satisfying the g-twisted orbifold boundary conditions

$$\begin{aligned} X^i(\tau, \sigma + 2\pi) &= \big[gX(\tau,\sigma)\big]^i = \big[\theta^k X(\tau,\sigma)\big]^i + \sum_{j=1}^{6} n_j e_j^i \\ X_{\mathrm{L}}^I(\tau, \sigma + 2\pi) &\equiv \big[gX(\tau,\sigma)\big]^I \bmod 2\pi\Lambda_{\mathrm{E}_8 \times \mathrm{E}_8} \\ &\equiv X_{\mathrm{L}}^I(\tau,\sigma) + 2\pi V_{k\text{-}n} \bmod 2\pi\Lambda_{\mathrm{E}_8 \times \mathrm{E}_8}. \end{aligned}$$

Given a second space group element $\tilde{g}$ of the centralizer of g, i.e. any element $\tilde{g} \in Z_g := \{\tilde{g} \in S : \tilde{g}g = g\tilde{g}\}$ commuting with g, one can apply it to a g-twisted string. This affects the boundary conditions in the fashion

$$\tilde{g}X(\tau, \sigma + 2\pi) = \tilde{g}gX(\tau,\sigma) = g\tilde{g}X(\tau,\sigma),$$

such that the state $\tilde{g}X$ still satisfies the g-twisted closed-string boundary conditions. Thus, for a state X of $\mathcal{H}_g$, the states $\tilde{g}X$ are also contained in $\mathcal{H}_g$ for any $\tilde{g} \in Z_g$.

Now suppose an element $h \in S \setminus Z_g$ not commuting with g is chosen, i.e. $hg \neq gh \iff hgh^{-1} \neq g$. Applying h to a g-twisted string state of $\mathcal{H}_g$ then yields

$$hX(\tau, \sigma + 2\pi) = hgX(\tau,\sigma) = (hgh^{-1})hX(\tau,\sigma) = \mathrm{Ad}_h(g)hX(\tau,\sigma).$$

Obviously, the state hX satisfies the boundary conditions associated with a hgh^{-1}-twisted string, which belongs to the Hilbert space $\mathcal{H}_{hgh^{-1}}$. In other terms, applying a non-commuting space group element effectively maps states from $\mathcal{H}_g$ to $\mathcal{H}_{hgh^{-1}}$. Since h and $h^n g(h^{-1})^n$ are also not commuting in general, repeated application of h to a g-twisted state yields a state contained in $\mathcal{H}_{h^n g(h^{-1})^n}$. If the non-commuting element $h \in S \setminus Z_g$ is idempotent, i.e. $h^n = e$ for some $n \in \mathbb{N}$, this chain becomes cyclic. This can be represented as follows:

$$\begin{tikzcd}
\mathcal{H}_g \arrow[loop above, "\tilde{g} \in Z_g"] \arrow[r, "h \in S \setminus Z_g"] & \mathcal{H}_{hgh^{-1}} \arrow[loop above, "\widetilde{hgh^{-1}} \in Z_{hgh^{-1}}"] \arrow[r, "h"] & \cdots \arrow[r, "h"] & \mathcal{H}_{\mathrm{Ad}_{h^n}(g)} \arrow[loop above, "\widetilde{\mathrm{Ad}_{h^n}(g)}"] \arrow[r, "\text{not idempotent}"] \arrow[lll, bend left, "\text{for idempotent } h\text{, such that } h^n = e"] & \mathcal{H}_{\mathrm{Ad}_{h^{n+1}}(g)} \arrow[loop above, "\widetilde{\mathrm{Ad}_{h^{n+1}}(g)}"] \arrow[r] & \cdots
\end{tikzcd}$$

Therefore, in order to construct S-invariant states, one has to consider linear combinations of the states contained in the Hilbert spaces of the form $h^j g(h^{-1})^j$ for $j \in \mathbb{N}$. For idempotent non-commuting elements h such a linear combination is finite—however, this will not be the case in general. Thus, one has to consider the elements of

$$\bigoplus_{j=1}^{\infty} \mathcal{H}_{\mathrm{Ad}_h(g)} = \bigoplus_{j=1}^{\infty} \mathcal{H}_{h^j g(h^{-1})^j}$$

for each constructing element g of the space group (i.e. for each twisted sector and the respective fixpoints) in order to arrive at the S-invariant states, that constitute the orbifold compactified particle spectrum. Nevertheless, carrying out all those tasks explicitly for a given orbifold requires quite some work.

CHAPTER 10

Semi-realistic compactifications

In this chapter a specific model of the $\mathbb{Z}_6$-II-orbifold is discussed, which was recently presented in [BHLR06a] and [BHLR06b]. The particular details of the model allow to recover most of the phenomenological properties of the standard model, which finally provides a semi-realistic compactification of the heterotic string. Some parts of this chapter are carried out in less detail as the previous exposition, as there seems to be no point in rewriting the original paper. Rather, the purpose of presenting this particular model is to show that the lengthy constructions of the preceding chapters actually lead to a model of physical value—aside from the general elegance of the entire string approach. Furthermore, the latter sections hint at several issues in the ongoing research of this subject, which are accompanied by more general remarks following in the next chapter.

10.1. Grand unified theories

Besides providing a remarkable accurate description of elementary particle physics, there is not much in favor of the standard model. It provides no explanation of the left-right asymmetry in weak interactions, requires renormalization techniques due to numerous divergences, depends on a large number of external parameters, etc. Furthermore, the fermionic matter content corresponds to a rather coincidental reducible representation of the underlying gauge group

$$G_{\mathrm{SM}} = \mathrm{SU}(3)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}.$$

Thus, not long after the standard model was established, physicists started investigating more elegant concepts. The central idea of any grand unified theory (GUT) is to use an enlarged simple Lie group as the fundamental gauge group of all matter, whose irreducible

	quarks $(\mathbf{3},\mathbf{2})_{\frac{1}{6}}$		**antiquarks** $(\bar{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}}$	$(\bar{\mathbf{3}},\mathbf{1})_{\frac{1}{3}}$	**leptons** $(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$		**antilep.** $(\mathbf{1},\mathbf{1})_{\mathbf{1}}$
gen. 1	up u_{L}	down d_{L}	$\overline{u_{\mathrm{R}}}$	$\overline{d_{\mathrm{R}}}$	electron e_{L}	$(\nu_e)_{\mathrm{L}}$	$\overline{e_{\mathrm{R}}}$
gen. 2	charm c_{L}	strange s_{L}	$\overline{c_{\mathrm{R}}}$	$\overline{s_{\mathrm{R}}}$	muon μ_{L}	$(\nu_\mu)_{\mathrm{L}}$	$\overline{\mu_{\mathrm{R}}}$
gen. 3	top t_{L}	bottom b_{L}	$\overline{t_{\mathrm{R}}}$	$\overline{b_{\mathrm{R}}}$	tauon τ_{L}	$(\nu_\tau)_{\mathrm{L}}$	$\overline{\tau_{\mathrm{R}}}$
el. chg. Q	$+\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$+\frac{1}{3}$	-1	0	$+1$
weak iso. I_z	$+\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	$+\frac{1}{2}$	0
hyperch. $\frac{Y}{2}$	$+\frac{1}{6}$	$+\frac{1}{6}$	$-\frac{2}{3}$	$+\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+1$
color ch. C	3	3	$\bar{3}$	$\bar{3}$	1	1	1

TABLE 10.1. Any of the three generations of spin-$\frac{1}{2}$ matter fermions of the standard model transforms under the particular $\mathrm{SU}(3)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$-representation $(3,2)_{\frac{1}{6}} \oplus (\bar{3},1)_{-\frac{2}{3}} \oplus (\bar{3},1)_{\frac{1}{3}} \oplus (1,2)_{-\frac{1}{2}} \oplus (1,1)_1$, where the subscript denotes the hypercharge $Y = 2(Q - I_z)$. There is an additional doublet of SU(2) for the Higgs boson and bosons for the three interactions described by the standard model.

representations split to the particular $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$-representation used in the standard model, see tab. 10.1. In 1974, Georgi and Glashow proposed the **SU(5)-GUT** model. As G_SM is a rank-4 Lie group and one requires complex representations for charge-conjugation, the choice of a suitable GUT groups is very limited, with SU(5) being the smallest possible candidate. As $\dim \mathrm{SU}(5) = 24$, it provides 24 gauge bosons, 12 of which are the familiar bosons mediating the standard model interactions, and 12 are additional bosons responsible for turning quarks into leptons, and vice versa. This particular property predicts a rapid proton decay not in agreement with experimental data—essentially, the SU(5)-GUT has been ruled out experimentally. However, the theory is able to explain the quantization of charge and predicts the Weinberg weak mixing angle.

One year later Georgi (and independently Fritzsch and Minkowski) investigated a model based on embedding SU(5) into a **SO(10)-GUT** model. However, the name is rather misleading, as one actually considers the representations of the universal covering group Spin(10). In an elegant way, an entire standard model matter generation fits into a single 16-plet of Spin(10), i.e.

$$\begin{aligned} \mathrm{Spin}(10) &\to G_\mathrm{SM} = \mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y} \\ 16 &\mapsto \underbrace{(3,2)_{\frac{1}{6}} \oplus (\bar{3},1)_{-\frac{2}{3}} \oplus (\bar{3},1)_{\frac{1}{3}} \oplus (1,2)_{-\frac{1}{2}} \oplus (1,1)_{1}}_{\text{standard model matter generation}} \oplus \overbrace{(1,1)_0}^{\text{sterile particle}}, \end{aligned}$$

with only a single, additional particle $(1,1)_0$. Since it has no electrical charge and does interact neither strongly nor weakly, it is effectively hidden (aside from the gravitational interaction, which is neglected in the standard model anyway) for all phenomenological purposes. For example, this could be a sterile right-handed neutrino, see [FM75]. Due to the revealed neutrino masses—which are problematic in the (unextended) standard model—such a right-handed neutrino is conceptually rather welcomed. The gauge coupling unification in this model happens at energies / masses $m_\mathrm{GUT} \approx 10^{16}\mathrm{GeV}$.

One also has to supply the (still hypothetical) Higgs boson, which in the standard model comes in the form of weakly interacting SU(2)-doublets $(1,2)_{\pm\frac{1}{2}}$. As a boson, it must stem from a representation of SO(10) itself. The fundamental representation of SO(10) splits like

$$10 \mapsto \underbrace{(1,2)_{\frac{1}{2}} \oplus (1,2)_{-\frac{1}{2}}}_{\text{needed Higgs doublets}} \oplus \overbrace{(3,1)_{-\frac{1}{3}} \oplus (\bar{3},1)_{\frac{1}{3}}}^{\text{undesired ``color Higgs'' triplets}},$$

yielding undesired strongly interacting "color Higgs" triplets besides the needed Higgs doublets. The Higgs triplets are responsible for the rapid proton decay predicted by any GUT, which seems to be completely suppressed in nature. One has to find a suitable mechanism to effectively get rid of those unphenomenological triplets (e.g. via high masses of order m_GUT), much like for the (as yet) unobserved superpartner particles required for supersymmetry.[a] This is the famous **doublet-triplet splitting problem**. Thus, in the view of the SO(10)-GUT, an entire generation of left-handed standard model matter can be contained in a complete 16-plet of Spin(10), whereas the Higgs boson is only described by a split multiplet.

There is another feature found in the SO(10)-GUT, as it can be regarded as the natural combination of two different extensions of the standard model. In addition to the SU(5)-GUT, it also contains the **left-right symmetric model**, which was originally investigated to get an understanding of the L-R-asymmetry observed in the electroweak model. Due to a symmetry breaking of the L-R-symmetric model's gauge group $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{B-L}$ at low energies, the parity-violating $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ electroweak theory is obtained. At high energies, the hypercharge quantum number Y associated with the $\mathrm{U}(1)_\mathrm{Y}$-symmetry is replaced

[a]In a certain sense, GUTs share a common problem with supersymmetry: An enlargement of the original $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$-gauge symmetry of the standard model provides a more elegant perspective with many simplification—which is of course no surprise, as any symmetry always simplifies a theory—but at the cost of introducing additional degrees of freedom (i.e. particles), which are not present in the original theory.

by the $B-L$ quantum number associated to the **baryon-lepton symmetry** $\mathrm{U}(1)_{\mathrm{B-L}}$, i.e. the difference between the baryon and lepton number. The additional Cartan generator of the rank-5 GUT gauge group SO(10) can thus be understood as generating the additional B-L symmetry.

10.2. Local grand unification in orbifold compactification

A higher-dimensional space-time allows for a number of improvements to circumvent the mentioned problems. The compactification of the 10d $\mathrm{E}_8 \times \mathrm{E}_8$-heterotic string naturally leads to the SO(10)-GUT model along a chain of embeddings and gauge symmetry breaking as depicted in fig. 8.4 on p. 101. In the previous chapter it was observed, that in orbifold compactifications the projection condition for S-invariant states is different for each twisted sector. It can be further influenced by the presence of suitable Wilson lines. Thus, for any fixpoint on the orbifold, there is a local gauge group for the twisted strings attached to it, and in general those gauge groups are different, as illustrated in fig. 10.1.

As learned from chap. 8, the gauge bosons remaining in the effective 4d theory are the (massless) zero modes of the original gauge bosons. The zero modes belong to those gauge bosons which satisfy all the different local projection conditions simultaneously, i.e. the remaining effective gauge group is the intersection of all the local gauge groups associated to the orbifold fixpoints. Thus, the standard model gauge group can be recovered from an orbifold compactification provided

$$G_{\mathrm{SM}} = \mathrm{SU}(3)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}} \subset \bigcap_{i=1}^{n_{\mathrm{fp}}} G_i^{\mathrm{local}}$$

holds, which of course requires $G_{\mathrm{SM}} \subset G_i^{\mathrm{local}}$ for all i. This intersection is well-defined, since all groups G_i^{local} are subgroups of $\mathrm{E}_8 \times \mathrm{E}_8$, such that there is no ambiguity of how to intersect the different groups. Often the GUT group SO(10) is considered instead of G_{SM}. Together with the supersymmetry of the heterotic string, the matter content of the (minimal supersymmetric) standard model (MSSM) can be obtained from an orbifold compactification of the heterotic string. More details on the local grand unification are found in [BHLR05b].

10.3. The $\mathbb{Z}_6$-II orbifold

A most promising model using the concept of local grand unification was recently presented by Buchmüller, Hamaguchi, Lebedev and Ratz, see [BHLR06b]. It is based on the toroidal Calabi-Yau orbifold $T^6/\mathbb{Z}_6$-II, which is related to the orbifold $T^6/(\mathbb{Z}_3 \times \mathbb{Z}_2)$, since $\mathbb{Z}_6$ is a cyclic group with a non-prime number of elements $6 = 3 \cdot 2$. The geometry and complex structure of the underlying 6-torus is specified by the simple roots of the Lie group $\mathrm{G}_2 \times \mathrm{SU}(3) \times \mathrm{SO}(4)$,

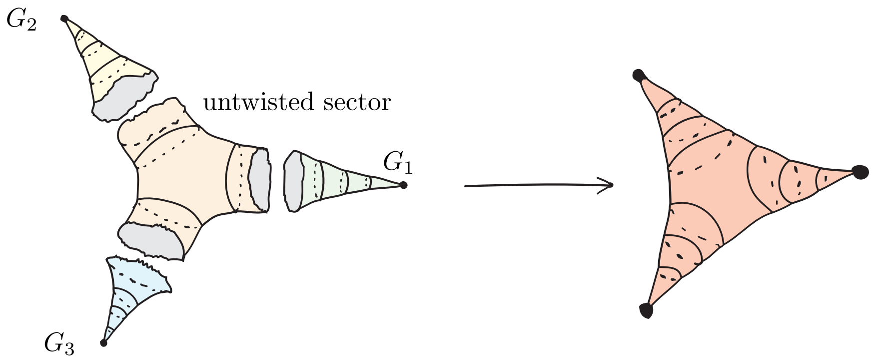


FIGURE 10.1. The $T^2/\mathbb{Z}_3$-orbifold splitted.

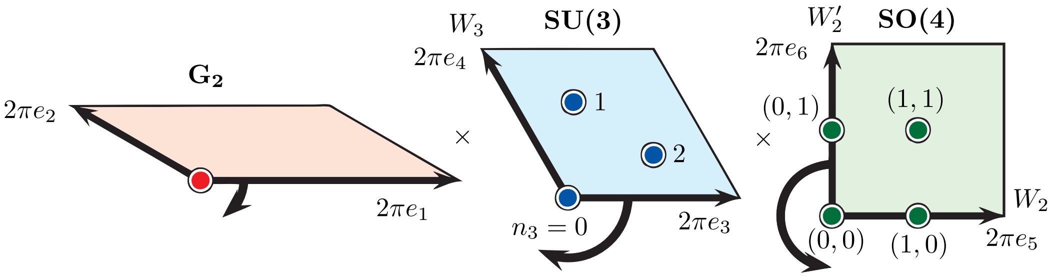


FIGURE 10.2. The $\mathrm{G}_2 \times \mathrm{SU}(3) \times \mathrm{SO}(4)$ root lattice with fixpoints as used in the definition of the $\mathbb{Z}_6$-II orbifold.

which are depicted in fig. 10.2, i.e. one uses the splitting

$$T^6 = T^6_{\mathrm{G}_2 \times \mathrm{SU}(3) \times \mathrm{SO}(4)} = T^2_{\mathrm{G}_2} \times T^2_{\mathrm{SU}(3)} \times T^2_{\mathrm{SO}(4)}.$$

The action of the point group $\mathbb{Z}_6$ is defined via the generating twist vector $v = \left(-\frac{1}{6}, -\frac{1}{3}, +\frac{1}{2}\right)$, which is the negative of the vector found in tab. 9.1 for the $\mathbb{Z}_6$-II orbifold. This particular choice of sign leads to left-handed states in the first twisted sector, see [BHLR06b, §3]. The action of the generator $\theta \in \mathbb{Z}_6$ then amounts to a clockwise $60°$ rotation in the G_2-plane, a clockwise $120°$ rotation in SU(3)-plane and a counter-clockwise $180°$ rotation in SO(4)-plane, as shown in fig. 10.2. In terms of the usual complex coordinates, the action of θ is represented as

$$\begin{array}{l} \theta : T^6_{\mathrm{G}_2 \times \mathrm{SU}(3) \times \mathrm{SO}(4)} \longrightarrow T^6_{\mathrm{G}_2 \times \mathrm{SU}(3) \times \mathrm{SO}(4)} \\ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{e}^{-\frac{\pi}{3}\mathrm{i}} z_1 \\ \mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}} z_2 \\ -z_3 \end{pmatrix} \end{array} \quad \rightsquigarrow \quad \left\{ \begin{array}{ll} 1 & \text{fixpoint on } \mathrm{G}_2\text{-plane}, \\ 3 & \text{fixpoints on SU(3)-plane}, \\ 4 & \text{fixpoints on SO(4)-plane}. \end{array} \right.$$

This yields 12 point-like fixpoints (0-dimensional conical singularities) and 12 corresponding twisted sectors. The fixpoints in the SU(3)-plane are denoted as $n_3 = 0, 1, 2$, whereas (n_2, n_2') refers to the four fixpoints in the SO(4)-plane, see fig. 10.2.

Since $\mathbb{Z}_6$ is a non-prime cyclic group, there are subgroups $\mathbb{Z}_2, \mathbb{Z}_3 \subset \mathbb{Z}_6$ which act as sub-twists on certain T^2-tori and allow for the presence of Wilson lines of order 2 and 3 in the respective coordinates. The subgroups arise naturally as $\mathbb{Z}_3 = \{\theta^2, \theta^4, \theta^6 = e\} \subset \mathbb{Z}_6$ and $\mathbb{Z}_2 = \{\theta^3, \theta^6 = e\} \subset \mathbb{Z}_6$. Since the $\mathbb{Z}_3$-generator acts explicitly as

$$\theta^2 : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}} z_1 \\ \mathrm{e}^{-\frac{\pi}{3}\mathrm{i}} z_2 \\ z_3 \end{pmatrix} \quad \rightsquigarrow \quad \left\{ \begin{array}{ll} 3 & \text{fixpoints on } \mathrm{G}_2\text{-plane}, \\ 3 & \text{fixpoints on SU(3)-plane}, \\ 1 & \text{fixed torus on SO(4)-plane} \end{array} \right.$$

it is clear, that the $\mathbb{Z}_3$-action keeps the SO(4)-plane invariant. Likewise, the explicit $\mathbb{Z}_2$-action

$$\theta^3 : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} -z_1 \\ z_2 \\ -z_3 \end{pmatrix} \quad \rightsquigarrow \quad \left\{ \begin{array}{ll} 4 & \text{fixpoints on } \mathrm{G}_2\text{-plane}, \\ 1 & \text{fixed torus on SU(3)-plane}, \\ 4 & \text{fixpoints on SO(4)-plane} \end{array} \right.$$

keeps the SU(3)-plane invariant. This information can be used to get a deeper understanding of the $\mathbb{Z}_6$-II-orbifold's geometry, as the $\mathbb{Z}_6$-fixpoints are equal to the intersection of the $\mathbb{Z}_2$- and $\mathbb{Z}_3$-fixpoints, see [BHLR06b, §3]. In particular, the $\mathbb{Z}_2$- and $\mathbb{Z}_3$-fixpoints in the G_2-plane can be used to show that—despite having only the trivial fixpoint under the $\mathbb{Z}_6$-action—the G_2-plane folds to a three-tipped "pillow". The same holds for the SU(3)-plane, whereas the SO(4)-plane has the geometry of a four-tipped "pillow" after identifying by the action of the point group $\mathbb{Z}_6$ on the $T^6_{\mathrm{G}_2 \times \mathrm{SU}(3) \times \mathrm{SO}(4)}$-torus. The explicit location of the fixpoints can be found in the original paper.

Recall from sec. 9.12, that all massless states in the first twisted sector obey the S-invariance condition. Since $T^6/\mathbb{Z}_6$-II is a non-prime orbifold, this is not the case for the other

sector	$\mathrm{G_2}$	SU(3)	SO(4)
$\mathfrak{T}^1$	A	3 fixpoints	4 fixpoints
$\mathfrak{T}^2$	A or B	3 fixpoints	bulk
$\mathfrak{T}^3$	A or C	bulk	4 fixpoints
$\mathfrak{T}^4$	A or B	3 fixpoints	bulk
$\mathfrak{T}^5$	A	3 fixpoints	4 fixpoints
untwisted $\mathfrak{T}^6 = \mathfrak{T}^0$	bulk	bulk	bulk

TABLE 10.2. Location of the twisted sectors on the T^2-planes, where A, B, C refers to the $\mathrm{G_2}$-“pillows” fixpoints.

twisted sectors and in particular not for the untwisted sector. However, due to the particular $\mathbb{Z}_2$- and $\mathbb{Z}_3$-subtwists of the considered $\mathbb{Z}_6$-II orbifold, the projection conditions for the S-invariant states are simplified, see [BHLR06b, §2.5.2]. One may introduce two Wilson lines $W_2 := A_5$ and $W_2' := A_6$ of order 2 for the two basis vectors spanning the $\mathbb{Z}_3$-invariant SO(4)-lattice. Likewise, a single order-3 Wilson line W_3 can be chosen in the $\mathbb{Z}_2$-invariant SU(3)-plane. Due to the particular way of naming the fixpoints (cf. fig. 10.2), the corresponding local gauge shift vector $V^{\text{fp}}_{k\text{-}n}$ for a space group element $(\ell, \theta^k) \in S$ associated to a fixpoint can be written as follows:

$$\begin{aligned}
&\text{SU(3)-plane:}\quad (\theta^k, ae_3 + be_4) \in S \quad &&\Longrightarrow\quad V^{\text{fp}}_{k\text{-}n} = kV + m_3 W_3 \text{ for } m_3 := a + b \bmod 3\\
& && \qquad\qquad = k(V + n_3 W_3)\\
&\text{SO(4)-plane:}\quad (\theta^k, \ell) \in S &&\Longrightarrow\quad V^{\text{fp}}_{k\text{-}n} = k(V + n_2 W_2 + n_2' W_2'),
\end{aligned}$$

each vector neglects the contribution of the respective other components. Twisted string states, which are closed by the action of $\theta^k \in P$ may either be localized at one of the respective fixpoints of each plane or may live in the bulk like the untwisted states. This is listed in tab. 10.2, where A, B, C refers to the three singularities of the $\mathrm{G_2}$-“pillow”. More precisely, A comes from the trivial fixpoint at the origin, B is one of the $\mathbb{Z}_2$-fixpoints on $\mathrm{G_2}$, and C arises from a $\mathbb{Z}_3$-fixpoint in the $\mathrm{G_2}$-plane.

10.4. Minimal Supersymmetric Standard Model from the Heterotic String

It remains to determine the gauge shifting vector V and the Wilson lines W_2, W_2' and W_3, such that the gauge group is broken down to the standard model gauge group $G_{\text{SM}} = \mathrm{SU(3)_C} \times \mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$ and a suitable matter content remains. More precisely, one aims to receive the minimal supersymmetric ($\mathcal{N} = 1$) extension of the standard model (MSSM) in the effective 4d field theory resulting from the $T^6/\mathbb{Z}_6$-II orbifold compactification. As it turns out, the breaking of the gauge group is a rather simple task besides a number of other requirements—in fact, the breaking of the gauge group was achieved decades ago in orbifold models not based on local grand unification, see [IKNQ87] and [CM88]. However, all those models suffer from other problems like exotic couplings and particles, etc.

One of the major issues in the present context is to get three generations of standard model matter based on an underlying local GUT structure. In the considered $T^6/\mathbb{Z}_6$-II orbifold model, the fermionic matter is gained from the 16-dimensional spin representation of Spin(10), as discussed before. Thus, the embedding of the space S has to be chosen in a way that allows for local SO(10)—or rather Spin(10)—gauge asymmetry at each orbifold fixpoint. One can show (see [BHLR06b, §5.1]) that there are only two suitable choices for for the gauge shifting vector V. One chooses the gauge embedding and Wilson lines to be

$$\begin{aligned} V &= \Big(-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{3},0,0,0,0,0\Big)\Big(\tfrac{17}{6},-\tfrac{5}{2},-\tfrac{5}{2},-\tfrac{5}{2},-\tfrac{5}{2},-\tfrac{5}{2},-\tfrac{5}{2},\tfrac{5}{2}\Big),\\ W_2 &= \Big(-\tfrac{1}{2},0,-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0,0\Big)\Big(\tfrac{23}{4},-\tfrac{25}{4},-\tfrac{21}{4},-\tfrac{19}{4},-\tfrac{25}{4},-\tfrac{21}{4},-\tfrac{17}{4},\tfrac{17}{4}\Big),\\ W_3 &= \Big(-\tfrac{1}{6},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{6},-\tfrac{1}{6},-\tfrac{1}{6},-\tfrac{1}{6},-\tfrac{1}{6}\Big)\Big(0,-\tfrac{2}{3},\tfrac{1}{3},\tfrac{4}{3},-1,0,0,0\Big), \end{aligned} \tag{10.1}$$

which satisfy the strong modular invariance conditions (9.6) from the last chapter. After the corresponding orbifold compactification of the heterotic $\mathrm{E}_8\times\mathrm{E}_8$-string, the remaining gauge group is

$$G_{\text{eff}} = \underbrace{\mathrm{SU}(3)\times\mathrm{SU}(2)}_{\text{from first }\mathrm{E}_8\text{-copy}}\times\underbrace{\big[\mathrm{SU}(4)\times\mathrm{SU}(2)'\big]}_{\text{from second }\mathrm{E}_8\text{-copy}}\times\mathrm{U}(1)^9,$$

A very peculiar issue of this model is the origin of the fermionic matter. Naturally, one would expect to use the three equivalent fixpoints of the SU(3)-plane to localize the three families of the standard model matter. This would at least provide a partial explanation for the three matter generations observed in experiment. However, this attempt was investigated in [BHLR05a] by the same authors prior to the $T^6/\mathbb{Z}_6$-II model, but it turned out to be unsatisfactory due to the appearance of chiral exotic states. Moreover, all investigated models based on toroidal $\mathbb{Z}_N$-orbifolds for $N\le 6$ seem to suffer from this problem. Thus, the three families of matter do not arise in the same fashion in the $T^6/\mathbb{Z}_6$-II model. One rather considers two equivalent families localized at two equivalent fixpoints in the SO(4)-plane—which is achieved by choosing only a single Wilson line on this plane instead of the possible two—and a third family arising from different twisted sectors:

$$\begin{aligned} &\text{equivalent families:} && 2\times 16_{\mathrm{Spin}(10)} \text{ of } \mathfrak{T}^1\\ &\text{“odd” third family:} && 1\times 16_{\mathrm{Spin}(10)} \text{ of } \mathfrak{T}^0,\mathfrak{T}^2,\mathfrak{T}^4. \end{aligned}$$

This particular construction also affects the respective Yukawa coupling of the third family and can thus be associated with the extremely heavy top-bottom family of the standard model.

Apart from the three phenomenologically desired matter families, a number of additional vector-like states is found, in particular a pair of Higgs doublets responsible for the mass-generating spontaneous symmetry breaking process. The spectrum of the $T^6/\mathbb{Z}_6$-II model presented in [BHLR06b] can thus be summarized to contain

$$3\times 16_{\mathrm{Spin}(10)} \text{ chiral matter} + \text{vectors},$$

with detailed tables of the states provided in the original paper. Further exotic matter found in the spectrum can be successfully decoupled below the GUT energy scale, see [BHLR06b, §5.5]. This also takes care of the additional gauge group factors $\mathrm{SU}(4)\times\mathrm{SU}(2)'$, which provide a hidden sector that may be responsible for a spontaneous breaking of the SUSY in terms of gaugino condensation. By preserving an approximate baryon-lepton symmetry for certain vacua—and breaking it for others—the model suppresses the problematic proton decay usually associated with GUTs. The supersymmetric vacua of the model, which has a large vacuum degeneracy, are discussed in [BHLR06b, §6] along with further phenomenological properties in its subsequent sections.

10.5. MSSM orbifold landscape

It turns out, that the particular choice (10.1) of the gauge shifting vector and the Wilson lines is only a single one of around $\mathcal{O}(10^{7\pm3})$ other embeddings. Furthermore, the same approach can be carried out using a local grand unification based on the GUT group E_6 instead of SO(10). This freedom of choices has stirred a (computer-supported) search program for semi-realistic orbifold compactifications of the heterotic string, which is currently carried out in the context of the corresponding “$T^6/\mathbb{Z}_6$-II orbifold landscape”. To date, around $\mathcal{O}(10^2)$ inequivalent models have been identified, all sharing similar properties to those previously discussed. The program is outlined in [LNR$^+$07] and the resulting models are found in the on-line database [LNR$^+$06].

Since orbifold compactifications can often be associated to singular points of a Calabi-Yau compactification moduli space, this raises the question whether the particular identified models belong to a rather "fertile" region of the general heterotic landscape, for further details see the next chapter.

CHAPTER 11

Outlook

In the preceding chapters the semi-realistic minimal supersymmetric standard model was derived from a certain compactification of the $E_8 \times E_8$-heterotic string on a singular space, which was developed from the fundamentals. The author tried to focus on the geometric properties and the appropriate mathematical description of the numerous symmetries encountered in string theory. One aim of this survey was to present a concise—but mathematically rather complete—introduction to the different types of compactification processes of the heterotic string with a highlight on orbifold compactifications. The discussed toroidal $\mathbb{Z}_6$-II orbifold model serves as one important example of what can be achieved along this road.

Despite the apparent success of this particular construction, the implied singular nature of space-time is rather counter-intuitive. As mentioned before, techniques of algebraic geometry allow to "blow-up" the singularities, such that a smooth manifold is obtained. This is explained in [Ref06, chp. 3] at great length for the types of singularities arising in the previously enumerated orbifolds. The resulting "crepant resolutions" (see [Joy00, §6.6] or [Joy07, §7.5]) are then smooth Calabi-Yau manifolds, which may yield the same string phenomenology as the original toroidal Calabi-Yau orbifold. However, in general, resolving just a single singularity is a minor task compared to resolving an entire orbifold—but it can be done in case of the toroidal orbifolds, see [LRSS06] and [Ref06]. In fact, resolved toroidal orbifolds are one of the few cases where the Calabi-Yau manifolds are quite well-understood. The $\mathbb{C}^3/\mathbb{Z}_6$-II prototype singularity is resolved in [Ref06, §A.3] or [LRSS06, §A.4], whereas the considered $T^6/\mathbb{Z}_6$-II orbifold with underlying $G_2 \times SU(3) \times SO(4)$ torus lattice is resolved in [LRSS06, §D.4]. However, this does not take into account the effect of the non-trivial choice of Wilson lines, which is crucial for the breaking of the internal gauge group and the fermion matter generations.

Unfortunately, Calabi-Yau compactifications of the heterotic string are also not satisfactory due to a problem generic to all compactifications based on the Kaluza-Klein mechanism. The arising geometric moduli of the compactification space can be interpreted as either massless fields or flat directions in an effective potential. This gives rise to a fifth interaction based on those moduli fields, for which there is no indication in nature. Therefore, in realistic string models based on Calabi-Yau compactification, one should find a mechanism to suppress the dynamics of the moduli fields. This is usually referred to as the problem of **moduli stabilization**.

One of the main achievements on string compactification in the 90's was the discovery of a generalized kind of Calabi-Yau compactification. Recall from chap. 8 that a vital step in deriving the conditions for $\mathcal{N} = 1$ SUSY in the 4d effective theory was requiring the 3-form field $\tilde{H}$ to vanish, which is usually called the **no-flux condition**. Considering the more general case of unrestricted $\tilde{H}$ leads to the so-called **flux compactifications** of string theory, which are reviewed in [BBS07, chp. 10]. An important fact about those compactifications is the existence of a non-trivial scalar potential for the moduli fields. Since any point of the moduli space is associated with a different vacuum due to the different space-time geometry, the local minima of this scalar potential are associated with (meta-)stable vacuum configurations. This gives rise to the recent notion of a vast **string landscape**, where the valleys are (meta-)stable vacua. Estimates of the number of such vacua go as high as 10^{500}. As yet, there are no supreme selection criteria known that would distinguish our own particular universe within this landscape—which is supposedly found in one of the deeper (more stable) of those valleys.

The previously mentioned "MSSM orbifold landscape" can be regarded as a certain subset of singular compactifications encountered in the heterotic landscape.

In 2003 a new way of obtaining stable string theory vacua with a non-vanishing cosmological constant was proposed in [KKLT03]. The greatest benefit of this KKLT model is the stabilization of the numerous moduli fields and the dilaton scalar, however, the particular construction only applies to the chiral type-IIB superstring. In this context, one considers a generalized notion of orbifold, where one allows for orientation-reversing actions of the point group. Such **orientifold compactifications** have profound implications for the resulting string phenomenology. In [Ref06, §7.5] the KKLT mechanism is considered for the case of resolved toroidal orbifolds, in particular for the $T^6/\mathbb{Z}_6$-II orbifold, albeit for the $\mathrm{SU}(2)\times\mathrm{SU}(6)$ torus lattice.

Thus, there are quite a few deep and interesting problems left open, which must be investigated in further research:

- For conceptual reasons, one should be able to find suitable Calabi-Yau manifolds that achieve the same phenomenological success as the orbifold compactification discussed in the last chapter. More generally, one may locate the region of MSSM orbifold compactifications within the heterotic landscape, which might help to identify the right vacua.
- The problem of moduli stabilization has to be solved for more general situations. There are some results in this direction, which extend the KKLT-approach to the heterotic string and non-geometric moduli. In the view of M-theory and the duality web (see [BBS07, chp. 8]), it would be satisfying to have a moduli stabilization procedure at hand, that is essentially independent of the particular string theory limit.
- Most of the presented results are indeed only shown to be true for the low-energy effective SUGRA approximation with massless fields. It remains to check, whether all results really extend to the full string theory and excited states.
- One has to prove that the results are still valid at the non-perturbative level. Since string theory still lacks a completely non-perturbative description, this check has to be done indirectly using D-branes, BPS states, etc.
- A completely background independent formulation of string theory would be highly appreciated, cf. sec. 7.2. In this context, one may also provide a more elegant (mathematical) proof of the finiteness of the higher perturbative orders, as mentioned in sec. 7.11.

Whether string theory will indeed live up to its supposed purpose of providing a completely unified theory of all known fundamental interactions or not, remains yet to be found out. Regardless, the apparent success of the presented orbifold model and the revealed geometric structure strongly hints at the conceptual validity of the entire string approach.

Part III

Appendices

APPENDIX A

Mathematical Foundations

The development of higher mathematical notions along the lines of a rather rigorous chain of reasoning requires some fundamentals that are introduced in this section for the sake of completeness. After introducing groups and vector spaces, the notion of a manifold is introduced. The geometric and physical properties of a manifold's tangent spaces and bundles are described in detail, as well as the (anti-)holomorphic splitting encountered for complex manifolds. Furthermore, a section is spend on issues concerning differential and complex structures and their importance to general physics. The classical groups are introduced as a general reminder and for later reference, as well as elementary aspects of group representation theory. This is extended in the later section to Lie groups, where the relation between the Lie group and Lie algebra is explained. In particular, the exponential mapping and its applicability domain is detailed, as well as the connection between Lie algebra and Lie group representations. In fact, this entire chapter contains no physical content but paves the way for the development of more advanced material in the main chapters. Of course, the entire section neglects to prove any of the mathematical statements, but provides plenty of references for further study.

A.1. Groups and homomorphisms

A general mapping $f : A \longrightarrow B$ between sets A and B is said to be **injective** if $x \neq y$ implies $f(x) \neq f(y)$ for all elements $x, y \in A$. The mapping f is **surjective** if for any given $b \in B$ there exists at least one element $a \in A$ such that $f(a) = b$. If both injectivity and surjectivity are satisfied, the mapping f is **bijective** (sometimes called "one-to-one"). In particular, bijective mappings are invertible and there exists an (inverse) mapping $g : B \longrightarrow A$ such that $f \circ g = \mathrm{Id}_B$ and $g \circ f = \mathrm{Id}_A$ are the identity mappings of the respective sets.

A **group** G is a set together with an associative composition operation $* : G \times G \longrightarrow G$, i.e. it satisfies $(a * b) * c = a * (b * c)$, that contains an (unique) **neutral** or **unit element** $e \in G$ satisfying $e * a = a * e = e$ and an (unique) **inverse element** $a^{-1} \in G$ for every $a \in G$. In addition, if commutativity $a * b = b * a$ holds, the group is called **abelian**. In more mathematical terms a group is a monoid with inverse elements. An abelian group is usually denoted with the addition symbol "+" for the composition operation, whereas a multiplicative notation hints at a non-commutative behavior of the group operation. Both the real numbers $\mathbb{R}$ and integers $\mathbb{Z}$ together with ordinary addition are groups, but the natural numbers $\mathbb{N}$ lack the inverse elements of addition, i.e. the negative integers.

A **subgroup** H of G is a subset $H \subset G$ containing the neutral element, such that in addition H is closed under the composition, i.e. contains the inverse elements. As the name suggests, a subgroup is itself a group, that inherits its composition operation from the parent group. A **trivial subgroup** just consists of the unit element alone.

Suppose $g \in G$ is a group element, such that consecutive operations of this element and its inverse generate the entire group G. For example, the group $(\mathbb{Z}, +)$ is generated by either $+1$ or -1, since every integer $n \in \mathbb{Z}$ can be written as $n = 1{+}1{+}\ldots{+}1$ or $n = (-1){+}(-1){+}\ldots{+}(-1)$. Elements with this property are called **generators** of the group.

Let G_1, G_2 be two groups and $G_1 \times G_2 = \{(g_1, g_2) : g_i \in G_i\}$ be the direct product as sets. The law of composition for the **direct group product**[a] is defined component-wise via

$$(x_1, x_2) * (y_1, y_2) := (x_1 y_1, x_2 y_2).$$

Then $G_1 \times G_2$ is also a group, whose neutral element is (e_1, e_2) and the inverse of (a, b) is (a^{-1}, b^{-1}). The direct product $G_1 \times \cdots \times G_n$ of n groups is defined completely analogously by n-tuples consisting of $x_i \in G_i$ and component-wise composition. This can be generalized even further to the direct product of arbitrary families $\{G_i\}_{i \in I}$ of groups with index set I.

Naturally, a sensible mapping between groups G, G' should preserve the group structure. A mapping $\varphi : G \longrightarrow G'$ is called a **group homomorphism** (or just homomorphism for short) if $f(ab) = f(a)f(b)$ holds for all elements $a, b \in G$. One easily proves $f(e) = e'$ and $f(a^{-1}) = f(a)^{-1}$ from this. A homomorphism is called an **isomorphism** if there exists an (inverse) homomorphism $\tilde{\varphi} : G' \longrightarrow G$ such that $\tilde{\varphi} \circ \varphi = \mathrm{Id}_G$ an $\varphi \circ \tilde{\varphi} = \mathrm{Id}_{G'}$ are the identity mappings of G and G' respectively. In particular, this implies that an isomorphism is a bijective mapping. If there exists an isomorphism between groups G, G' it is usually denoted $G \cong G'$ for short. A homomorphism $\psi : G \longrightarrow G$ from a group G into itself is called an **endomorphism**. Note that an endomorphism does not need to be invertible, i.e. it is in general neither bijective nor an isomorphism. A bijective endomorphism, i.e. an isomorphism of the group into itself, is called an **automorphism**. Together with two additional types of homomorphism, all these statements can be summarized as follows:

homomorphism		$G \longrightarrow H$	$\in$	$\mathrm{Hom}(G, H)$
monomorphism:	injective	$G \hookrightarrow H$	$\in$	$\mathrm{Mono}(G, H)$
epimorphism:	surjective	$G \twoheadrightarrow H$	$\in$	$\mathrm{Epi}(G, H)$
isomorphism:	bijective	$G \xrightarrow{\cong} H$	$\in$	$\mathrm{Iso}(G, H)$
endomorphism:		$G \longrightarrow G$	$\in$	$\mathrm{End}(G)$
automorphism:	bijective	$G \xrightarrow{\cong} G$	$\in$	$\mathrm{Aut}(G)$

Let $f : G \longrightarrow G'$ be a group homomorphism and $e' \in G'$ the unit element of the target space. The **kernel** of f is defined to be the subset of G consisting of all elements $g \in G$ such that $f(g) = e'$, denoted "$\ker f$" for short. The set "$\mathrm{im}\, f$" is called the **image** of the mapping and consists of $f(g)$ for all $g \in G$. Both $\ker f \subset G$ and $\mathrm{im}\, f \subset G'$ are subgroups of their respective parent sets.

A subgroup $H \subset G$ induces a **left coset** $aH := \{ah : h \in H\} \subset G$ and **right coset** $Ha := \{ha : h \in H\}$ for any element $a \in G$. In particular, any element of G is contained in a left coset and a right coset. For commutative groups, both notions coincide. A subgroup H that satisfies the condition $gHg^{-1} = H$ in the set-theoretic sense for all $g \in G$ is called **normal**, which in particular implies $gH = Hg$, thus left and right cosets coincide in this case, too. The **factor group** or **coset space** G/H is then defined as the group of all (left or right) cosets of G, where the group operation is induced via

$$\begin{aligned} *' : G/H \times G/H &\longrightarrow G/H \\ (aH, bH) &\mapsto aH *' bH := abH. \end{aligned}$$

Suppose the set R is equipped with two laws of composition, called addition and multiplication, such that both the associative multiplication has a unit element and $(R, +)$ is commutative group. Furthermore, if both compositions are connected by distributivity, then R is called a **ring**. In general, the multiplication operation of a ring in non-commutative. A subset $I \subset R$ is called a **right ideal**, if $(I, +)$ is a subgroup of $(R, +)$ and $xr \in I$ for all $x \in I$ and all $r \in R$, the left ideal is defined analogously. A **field** F is a commutative ring where inverse elements of the multiplication are present and the neutral elements of the addition and multiplication do not coincide, i.e. $0 \neq 1$. A concrete example for a ring are the integers $\mathbb{Z}$,

[a]In the physical literature one sometimes finds the direct product of groups denoted as $G_1 \otimes G_2$ or $G_1 \oplus G_2$. While the meaning of this is of course clear in the specific context, the notation is quite wrong—there is no tensor product or direct sum defined for groups as it is for vector spaces.

which lack multiplicative inverses. In contrast, the rational numbers $\mathbb{Q}$ possess multiplicative inverses, and thus constitute a field.

In the context of normal subgroups, there is another type of product. Given a group G, a normal subgroup $N \subset G$, a subgroup $H \subset G$ and a homomorphism $\varphi : H \longrightarrow \operatorname{Aut}(N)$ usually defined by conjugation, i.e. $\varphi_h(n) := hnh^{-1}$, then the **semidirect product** of N and H via φ is the group defined as the cartesian product $N \times H$ with the product

$$(n_1, h_1) * (n_2, h_2) := \big(n_1 \varphi_{h_1}(n_2), h_1 h_2\big).$$

This is usually denoted as $N \ltimes_\varphi H := (N \times H, *)$, where the index φ is usually dropped. Note that in the case of $N, H \subset G$ both being normal subgroup, the semidirect product is not symmetric, i.e. $N \ltimes H \neq H \ltimes N$. Further elaborations of all these elementary notions are found in any algebra textbook, e.g. [Lan93, chp. 1].

A.2. Vector and Hilbert spaces

A **module** over the ring R, called an R-module for short, is an abelian group M under addition $+ : M \times M \longrightarrow M$ together with a mapping $\cdot : R \times M \longrightarrow M$. This **scalar multiplication** satisfies a special form of associativity $(\lambda\kappa) \cdot m = \lambda \cdot (\kappa \cdot m)$ for $\lambda, \kappa \in R$, $m \in M$, and distributivity

$$\lambda \cdot (m + n) = \lambda \cdot m + \lambda \cdot n$$

holds. The unit element under addition is $0 \in M$ and the multiplicative unit element $e \in R$ satisfies $e \cdot m = m$ for all $m \in V$. Furthermore, if the ring R is in fact a field F, then the associated module is called a **vector space**, i.e. a vector space is a module over a field. Conversely, a module is a vector space where the scalar multiplication is not commutative ($a \cdot b \cdot v \neq b \cdot a \cdot v$ for $a, b \in R$ and $v \in V$) and lacks inverses. $\{0\}$ refers to the (0-dimensional) vector space consisting solely of the zero.

The intersection of all subspaces containing a given set of vectors $v_1, \dots, v_n \in V$ is called its **span** and denoted "span$(v_1, \dots, v_n)$". If no vector v_i can be removed without changing the span, the set is said to be **linearly independent**. A linearly independent set whose span is V is called a **basis** for V. Using the axiom of choice or Zorn's lemma, it can be proved that every vector space has a basis. The number of vectors in a basis is called its **dimension**. In particular, a basis allows to represent every element $v \in V$ in a unique way as a linear combination of basis elements.

Note that one cannot attribute the notion of a basis to a module. If M is a R-module, a family $\{x_i\}_{i \in I}$ of elements $x_i \in M$ is called a **generating set** if $M = \sum_{i \in I} R x_i$ holds. If $\{x_i\}_{i \in I}$ can be chosen as a finite set of elements, M is called **finitely generated**. Moreover, a generating set is called **free** if its elements are linearly independent. A freely generated module is called **free *R*-module** for short, where the generating set $\{x_i\}_{i \in I}$ is often referred to as its **basis** due to the obvious similarity of the respective notion for vector spaces.

An **algebra** over the field F, called F-algebra for short, is a vector space $\mathcal{A}$ over F equipped with a binary multiplication operation, i.e. for $a, b \in \mathcal{A}$ there exists a operation such that $ab \in \mathcal{A}$. One might further restrict to **commutative algebras** where $ab = ba$. It is important to notice that there need not be any inverse elements be present in the algebra. A simple example of an algebra are the quadratic matrices over the field F, where the algebra multiplication is just the ordinary matrix multiplication.

A **module homomorphism** is a map $f : M \longrightarrow M'$ of one R-module M into another R-module M', such that $f(ax) = af(x)$ is satisfied for all $a \in R$ and $x \in M$. If one wishes to distinguish the ring R, this is also called a ***R*-homomorphism**.

Mappings $f : V \longrightarrow W$ between vector spaces V, W over the same field F are called **linear transformations** or **vector space homomorphisms** if the vector addition and the scalar multiplication is preserved, i.e. those mappings are linear $f(\lambda v + \kappa w) = \lambda f(v) + \kappa f(w)$. The set of linear transformations is denoted $\operatorname{Hom}_F(V, W)$ and is itself a vector space over F. The **kernel** of a linear transformation $f : V \longrightarrow W$ is defined to be all the vectors $v \in V$

which are mapped to $0 \in W$, likewise the **image** consists of the vectors $f(v) \in W$ for all $v \in V$. When bases for both V and W are chosen, every linear map can be expressed in terms of components as matrices.

Just like for groups, a linear transformation $f : V \longrightarrow W$ is an **isomorphism** if there exists an (inverse) linear transformation $f^{-1} : W \longrightarrow V$ such that $f^{-1} \circ f$ and $f \circ f^{-1}$ are the identity transformations on the respective vector spaces. Similarly, an **endomorphism** (sometimes also called **linear operator**) is a vector space homomorphism of V into itself. The invertible endomorphisms of a vector space V are called **automorphisms**. The set of automorphisms of vector spaces is denoted both $\mathrm{Aut}(V)$ and $\mathrm{GL}(V)$, which itself is a group under composition of mappings.

Let V be a n-dimensional real vector space with two **ordered bases** $\mathcal{B}_1 := (e_1, \ldots, e_n)$ and $\mathcal{B}_2 := (e'_1, \ldots, e'_n)$. Then there exists a unique linear automorphism $T \in \mathrm{Aut}(V)$ which maps the basis vectors $e_i \mapsto e'_i$, that is given in terms of inner products of the respective basis vectors. The two bases are **consistently oriented** if the determinant of the transformation T is positive. This provides an equivalence relation on the set of all ordered bases of V. An **orientation** on the vector space V is the assignment of "+1" to one of the two resulting equivalence classes and "−1" to the other.

Let V and W be two vector spaces over the same field F. From the (set theoretic) **cartesian product** $V \times W$ a new vector space $V \oplus W$, the **direct sum** of the vector spaces V and W, can be constructed by defining the vector space operations component-wise to be

$$\begin{aligned}(v_1, w_1) + (v_2, w_2) &:= (v_1 + v_2, w_1 + w_2),\\ \lambda \cdot (v, w) &:= (\lambda v, \lambda w).\end{aligned}$$

For the direct sum the subspace isomorphisms $V \times \{0\} \cong V$ and $\{0\} \times W \cong W$ are obvious.

Let $\{e_i\}$, $\{e'_j\}$ be the respective bases of V and W. The **tensor product** or **direct product** $V \otimes_F W$ over the field F is the span of the vectors $\{e_i \otimes e'_j\}$, where the vector space structure becomes immediate with the relations

$$\begin{aligned}(v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w,\\ v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2,\\ \lambda \cdot v \otimes w &= (\lambda v) \otimes w = v \otimes (\lambda w).\end{aligned}$$

Tensor products have an **universal property**, that gives an alternative understanding independent of a particular choice of basis elements: Let V, W be two vector spaces over the field F. Then there is a vector space $V \otimes_F W$ over the field F and a bilinear mapping $\eta : V \times W \longrightarrow V \otimes_F W$ with the following universality property: for any vector space U and bilinear mapping $\xi : V \times W \longrightarrow U$ there exists a linear mapping $\tilde{\xi} : V \otimes_F W \longrightarrow U$ such that $\xi = \tilde{\xi} \circ \eta$, i.e. the diagram

$$\begin{tikzcd} V \times W \arrow[d, "\eta \text{ bilinear}"'] \arrow[drr, "\xi \text{ bilinear}"] \arrow[dr, phantom, "\circlearrowleft"] & & \\ V \otimes_F W \arrow[rr, "\tilde{\xi} \text{ linear}"'] & & U \end{tikzcd}$$

can be made commutative for any U and ξ. The dimensions of the constructed vector spaces are $\dim V \oplus W = \dim V + \dim W$ for the direct sum and $\dim V \otimes_F W = \dim V \cdot \dim W$ for the direct product.[b]

Graded rings, **graded modules** and **graded algebras** are direct sums of the respective algebraic constructions equipped with a certain multiplication: Let $R = \bigoplus_{i=1}^{N} R_i$ be a graded ring, then there exist mappings $R_i \times R_j \longrightarrow R_{i+j \bmod N}$ for any $1 \leq i, j \leq N$, likewise for graded

[b]Note the importance of the subscript "F" at the tensor product $V \otimes_F W$, which indicates that it is a tensor product as vector spaces over F. Due to $\mathbb{C} \cong \mathbb{R}^2$ as $\mathbb{R}$-vector spaces, it follows $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$, but $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.

modules and graded algebras. In particular, $\mathbb{Z}_2$-graded algebras are called **superalgebras**, which are investigated in chap. 6 in much greater detail.

An **inner product** or **scalar product** on the vector space V over the field F is a positive-definite (that is $\langle v, v\rangle > 0$ for all vectors $0 \neq v \in V$ and $\langle v, v\rangle = 0$ if and only if $v = 0$)

for $F = \mathbb{R}$: bilinear form $\langle .,.\rangle : V \times V \longrightarrow \mathbb{R}$ where $\langle \alpha v, \beta w\rangle = \alpha\beta\langle v, w\rangle$

for $F = \mathbb{C}$: sesquilinear form $\langle .,.\rangle : V \times V \longrightarrow \mathbb{C}$ where $\langle \alpha v, \beta w\rangle = \alpha\bar{\beta}\langle v, w\rangle$.

A vector space V equipped with an inner product is called an **inner product space** or **pre-Hilbert space**. An inner product canonically induces a quadratic form $q(v) := \langle v, v\rangle$, which in turn gives rise to a canonical **norm** $\|v\| := \sqrt{q(v)}$. If a pre-Hilbert space is complete with respect to this norm, i.e. every Cauchy sequence converges, it is called a **Hilbert space**. A Hilbert space is called **separable** if it has a countable orthonormal basis.

Let $\mathcal{H}$ be a complex Hilbert space and $A : \mathcal{H} \longrightarrow \mathcal{H}$ a linear operator acting on $\mathcal{H}$. The (Hermitian) **adjoint operator** is defined by the property $\langle Ax, y\rangle = \langle x, A^\dagger y\rangle$. An **unitary operator** $U : \mathcal{H} \longrightarrow \mathcal{H}$ is a (bounded) linear operator satisfying $U^\dagger U = UU^\dagger = \mathrm{Id}$. The group of unitary operators on $\mathcal{H}$ is denoted by $\mathrm{U}(\mathcal{H})$.

A.3. Point-set topology

Before the introduction of manifolds, a little elementary point-set topology should be given in advance. First, a **topology** $\mathcal{T}$ for the set X has to be specified by declaring subsets $U \subset X$ as **open** in a compatible way, i.e. $\mathcal{T}$ is a collection of subsets of X—which are defined to be open—satisfying the following axioms: the empty set $\emptyset$, the entire set X as well as arbitrary unions plus finite intersections of open sets have to be open as well, i.e. for any collection $\{U_\lambda\}$ of open sets $U_\lambda \in \mathcal{T}$ holds

$$\emptyset, X \in \mathcal{T}, \qquad \bigcup_{\lambda\in J} U_\lambda \in \mathcal{T}, \qquad \bigcap_{n=1}^{N} U_n \in \mathcal{T},$$

where J is an arbitrary (possibly uncountable infinite) index set. Equivalently, one can specify a set of **closed** subsets of M which one gets by taking complements of open sets, thus the empty set and the entire set M are always both open and closed. More details are found in any elementary point set topology textbook, e.g. [Mun00, chp. 2]. In general, a set X together with a topology $\mathcal{T}$ is called a **topological space**.

Suppose a subset $\mathcal{T}' \subset \mathcal{T}$ can be chosen in a way, such that all open sets in $\mathcal{T}$ can be written as unions of sets from $\mathcal{T}'$. Such a $\mathcal{T}'$ is called a **base of topology** and many topological properties can be reduced to their respective bases. In particular, a topological space is called **second-countable** if its topology possesses a countable base of topology.

Furthermore, a topological space $(X, \mathcal{T})$ is called **Hausdorff** if any two points $x_1, x_2 \in X$ have disjoint open neighborhoods $U_1, U_2 \in \mathcal{T}$. This is quite a strong requirement for topological spaces. Suitable counterexamples are explained in [SS51, chp. 2] along with many other counter-intuitive curiosities of topology. A space is called **connected** if it cannot be split in a disjoint union of two or more nonempty open spaces.

The last property taken advantage of is compactness. A subset $\mathcal{U} \subset \mathcal{T}$ is called a **covering** of X if the union of all open sets in $\mathcal{U}$ contains the entire space X, i.e. $\bigcup_{U\in\mathcal{U}} U \supset X$. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{U}$ a covering of X, then $\mathcal{U}$ in general can consist of infinite many open sets. The space X is called **compact** if any covering $\mathcal{U} \subset \mathcal{T}$ contains a finite subset $\mathcal{U}' \subset \mathcal{U}$ that still covers X, i.e. any compact topological space can be covered with just finite many open subsets. The theorem of Heine-Borel provides that any subset $U \subset \mathbb{R}^n$ is compact if its closed and bounded.

Now let $f : X \longrightarrow Y$ be a function between two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. Intuitively, the notion of continuity implies that points, which are nearby in X are mapped to nearby points in Y. With respect to the specified topologies, this idea is formalized as follows:

The function f is **continuous** if the preimage $f^{-1}(U)$ of any open set $U \in \mathcal{T}_Y$ is an open set of X, i.e. the preimage f^{-1} maps elements of $\mathcal{T}_Y$ to elements of $\mathcal{T}_X$.

Let Y be an arbitrary set. A **metric** on Y is a positive, symmetric mapping $d : Y \times Y \longrightarrow \mathbb{R}$ satisfying the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$ for any three elements $x, y, z \in Y$. An **open ball** with respect to the metric d is the set

$$B_\varepsilon(x_0) := \big\{\text{all points } x \in Y \text{ where } |x - x_0| < \varepsilon \text{ for any } \varepsilon \in \mathbb{R}\big\}.$$

A pair (Y, d) is called a **metric space** and has a natural topology induced by the metric as follows: A subset $U \subset Y$ is called open, if it consists of a (arbitrary) union of open balls. Thus, any metric space is also a topological space. In particular, $\mathbb{R}^n$ has a natural topology induced by the Euclidean metric.

A stronger notion of connectedness (cf. four paragraphs above) is that of a **path-connected** space X, where any two points can be joined by a continuous path $\gamma : [0,1] \longrightarrow X$ and $[0,1]$ has the topology induced by restriction of $\mathbb{R}$. However, both notions coincide on manifolds, thus there is not need to distinguish between them. A path-connected space is called **simply-connected** if any closed curve can be continuously deformed into a point, e.g. $\mathbb{R}^2$ is simply-connected whereas $\mathbb{R}^2 \setminus \{0\}$ is not.

A.4. Universal coverings

Let X and C be topological spaces. A continuous surjective mapping $\pi : C \longrightarrow\!\!\!\!\rightarrow X$ is called a **covering map** if for every $x \in X$ there exists an open neighborhood $U \subset X$ such that the preimage $\pi^{-1}(U)$ is a union of mutually disjoint open sets each of which is mapped homeomorphically (continuous with a continuous inverse) onto U by π. Given such a map, the space C is called the **covering space** (or **cover** for short) of X. The preimage of a point $x \in X$ under a covering map is called the **fiber** over x. The real line $\mathbb{R}$ as a spiral hovering over the unit circle in $\mathbb{C}$ provides a classic example of a covering space, wherein the covering map is $\mathbb{R} \ni t \mapsto \mathrm{e}^{\mathrm{i}t} \in S^1 \subset \mathbb{C}$.

Let $\tilde{\pi} : \tilde{C} \longrightarrow\!\!\!\!\rightarrow X$ be a cover of X. This covering is called **universal** if for any other cover $\pi : C \longrightarrow\!\!\!\!\rightarrow X$ with connected covering space C there exists a covering map $f : \tilde{C} \longrightarrow\!\!\!\!\rightarrow C$ satisfying $\pi \circ f = \tilde{\pi}$:

$$\begin{tikzcd}
\tilde{C} \arrow[rr, two heads, "f"] \arrow[dr, two heads, "\substack{\text{covering}\\ \text{map } \tilde{\pi}}"'] & \arrow[d, phantom, "\circlearrowright"] & C \arrow[dl, two heads, "\substack{\text{covering}\\ \text{map } \pi}"] \\
& X &
\end{tikzcd}$$

Thus the universal covering of X also covers all connected covers of X. In particular, it is unique in the sense that for two universal coverings $\tilde{\pi}_1$, $\tilde{\pi}_2$ there exists a homeomorphism f between the respective universal covering spaces such that $\tilde{\pi}_1 \circ f = \tilde{\pi}_2$. A space must be path-connected, locally path-connected and semi-locally simply connected to have a universal covering. Details of this construction are found in every elementary topology text, e.g. [Mun00, chp. 13].

A.5. Topological, differential and complex manifolds

In simple terms, a n-dimensional manifold is a topological space that locally looks like a region of $\mathbb{R}^n$ or $\mathbb{C}^n \cong \mathbb{R}^{2n}$, often together with some additional properties. At this point only topological manifolds, differentiable manifolds and complex manifolds are discussed. In the next chapters manifolds of special type—particularly Kähler and Calabi-Yau manifolds—will be introduced.

The simplest type of structure one can require to extend to manifolds is continuity, which is naturally defined for mappings on $\mathbb{R}^n$. A n-dimensional **topological manifold** is a topological second-countable Hausdorff space M which is locally homeomorphic to open subsets of $\mathbb{R}^n$. That is, for every point $p \in M$ there exists an open neighborhood $U \subset M$, an open set

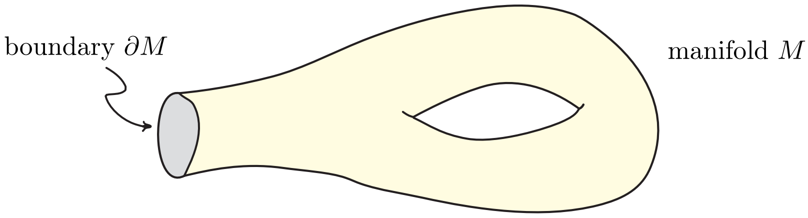


FIGURE A.1. A 2-dimensional manifold and its boundary.

$V \subset \mathbb{R}^n$ and a continuous mapping $\psi : U \longrightarrow V$ together with a continuous inverse mapping $\psi^{-1} : V \longrightarrow U$. Such a pair (ψ, U) is known as a **chart** of the manifold and a collection of charts that covers the entire manifold is called an **atlas** $\mathcal{A}$.

Let $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$ be the real numbers ≥ 0, where the index "0" indicates $0 \in \mathbb{R}_0^+$. Then one could also consider topological spaces that look locally like $\mathbb{R}_0^+ \times \mathbb{R}^{n-1}$, which gives rise to the notion of a **manifold with boundary**. The boundary itself is a $(n-1)$-dimensional submanifold without boundary and usually denoted as ∂M. Compact manifolds without boundary are usually called **closed manifolds**—imagine a torus for example. Most statements for boundaryless manifolds are also valid for manifolds with boundary, thus there will be no explicit distinction made in the further expositions.

A **continuous function** $f : M \longrightarrow N$ between topological manifolds is defined via the respective topologies. Equivalently, the composition $\psi_N \circ f \circ \psi_M^{-1} : V_M \longrightarrow V_N$ as in

$$\begin{tikzcd}
U_M \arrow[r, "f"] \arrow[d, "\psi_M"', "\approx"] \arrow[dr, phantom, "\circlearrowleft"] & U_V \arrow[d, "\psi_N", "\approx"'] & \text{(on manifold)} \\
V_M \arrow[r, dashed, "\psi_N \circ f \circ \psi_M^{-1}"'] & V_N & \text{(on } \mathbb{R}^n\text{)}
\end{tikzcd}$$

must be a continuous mapping between open sets of $\mathbb{R}^n$ for any pair of charts (ψ_M, U_M) and (ψ_N, U_N) of the respective manifolds.

Suppose a point $p \in M$ is contained in the overlap of two charts, i.e. there are charts (ψ_i, U_i) and (ψ_j, U_j) with $p \in U_i \cap U_j$. The composition $\psi_j \circ \psi_i^{-1} : V_i \longrightarrow V_j$ is a continuous mapping between open subsets of $\mathbb{R}^n$, called the **transition function** between the respective charts, which maps $\psi_i(p)$ to $\psi_j(p)$. If all transition functions of a given atlas are smooth (infinite often differentiable as mappings between $\mathbb{R}^n$ subsets, i.e. C^∞-differentiable) one has a **differentiable** or **smooth manifold**. A mapping $g : M \longrightarrow N$ between differentiable manifolds is called a **differentiable** or **smooth function** if any composition $\psi_N \circ g \circ \psi_M^{-1}$ is a smooth mapping on $\mathbb{R}^n$-subsets.

One can impose further restrictions on an even-dimensional manifold. Due to the canonical vector space isomorphism $\mathbb{R}^{2n} \cong \mathbb{C}^n$ the transition functions can be understood as functions $\mathbb{C}^n \longrightarrow \mathbb{C}^n$. A n-dimensional **complex manifold** is a $2n$-dimensional topological manifold where all transition functions are holomorphic in the usual sense. In particular, a connected 1-dimensional complex manifold is called a **Riemann surface**. One usually writes $\dim_{\mathbb{C}} M = n$ or $\dim_{\mathbb{R}} M = 2n$ to indicate the complex or real dimension of complex manifolds. A mapping $f : M \longrightarrow N$ is called **holomorphic** or **analytic** if the compositions $\psi_N \circ f \circ \psi_M^{-1}$ are holomorphic mappings $\mathbb{C}^m \longrightarrow \mathbb{C}^n$ for any pairs of charts.

In essence, using pairs of charts one pulls a mapping between the manifolds locally back to open subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ and requires continuity, differentiability or holomorphy as $\mathbb{R}^n$- or $\mathbb{C}^n$-mappings. The requirement of the transition functions to be continuous, smooth or holomorphic mappings ensures that the local regions patch together in a compatible way. This

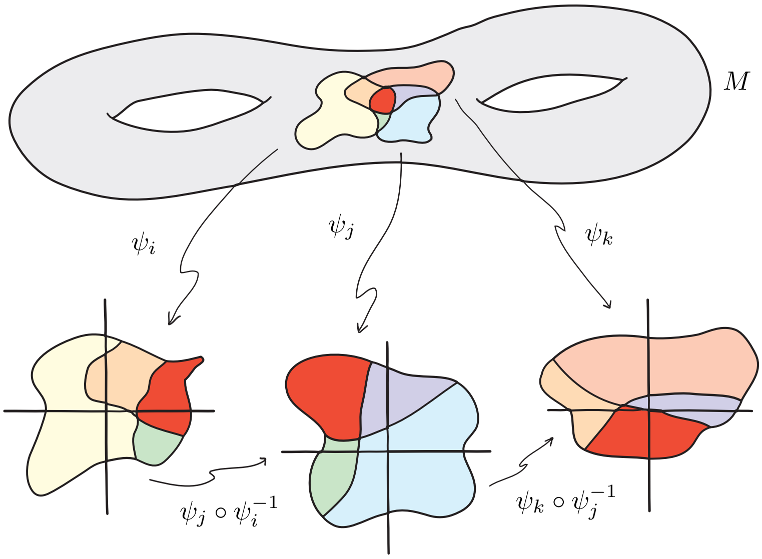


FIGURE A.2. A 2-dimensional manifold with a threefold chart overlap.

chain of imposed restrictions can be summarized as follows:

$$\begin{matrix}\text{topological manifold} \\ \text{(continuity)}\end{matrix} \supset \begin{matrix}\text{differentiable manifold} \\ \text{(differentiability)}\end{matrix} \supset \begin{matrix}\text{complex manifold} \\ \text{(holomorphy)}\end{matrix}.$$

To distinguish between different manifolds of the same type one has to introduce certain equivalence notions. Since topological manifolds provide a notion of continuity, two such manifolds M and N are called equivalent if there exists a **homeomorphism** $f : M \longrightarrow N$, i.e. a continuous mapping with continuous inverse mapping relative to the topologies of M and N.[c] Two differentiable manifolds are equivalent if there exists a **diffeomorphism** $g : M \longrightarrow N$, i.e. a differentiable mapping together with a differentiable inverse mapping. The space of diffeomorphisms $M \xrightarrow{\approx} N$ is usually denoted $\mathrm{Diff}(M, N)$ and $\mathrm{Diff}(M)$ refers to diffeomorphisms $M \xrightarrow{\approx} M$. The spaces M and N are equivalent as complex manifolds if there exists a **biholomorphic** mapping $h : M \longrightarrow N$, i.e. a holomorphic mapping with an inverse holomorphic mapping. Certain properties of topological spaces that are invariant under homeomorphisms are called **topological invariants**, e.g. all deformations of the sphere have the same topological invariants, but not a sphere and a torus as they have clearly different topologies and thus are not homeomorphic.

A.6. Differential and complex structures

If all transition functions of an atlas $\mathcal{A}$ are differentiable, $\mathcal{A}$ is called a differentiable atlas. Topological and holomorphic atlases are defined accordingly. Two (topological, differentiable or holomorphic) atlases $\mathcal{A}$, $\mathcal{A}'$ for a manifold M are called **compatible** if the union $\mathcal{A} \cup \mathcal{A}'$ of charts yields an atlas of the same type, i.e. charts of $\mathcal{A}$ and charts of $\mathcal{A}'$ are related by continuous, smooth or holomorphic transition functions.

The set-theoretic relation "$\subset$" provides a partial ordering between atlases of the same type. An atlas $\tilde{\mathcal{A}}$ is **maximal**, if any compatible atlas $\mathcal{B}$ of the same type is already a subset $\mathcal{B} \subset \tilde{\mathcal{A}}$. For any given atlas $\mathcal{A}$ there is exactly one unique maximal atlas $\tilde{\mathcal{A}} \supset \mathcal{A}$, called a **topological**, **differentiable** or **complex structure** of the respective manifold. Homeomorphisms,

[c] Of course, homeomorphisms are well-defined even for topological spaces. Note that there is no connection between the terms "homomorphism" and "homeomorphism".

diffeomorphisms and biholomorphisms provide an equivalence relation of their respective type of atlases.

It is important to note that a manifold may possess many inequivalent differential structures.[d] More precisely, up to diffeomorphisms there is only a single differentiable structure for manifolds of dimension ≤ 3 as shown by Randon. Kirby and Siebenman proved that the number of differential structures for manifolds of dimension ≥ 5 is finite. The case of 4-dimensional manifolds is most puzzling, as there may be uncountable infinite many differentiable structures.[e] In particular, the mathematical world was stunned, when Taubes—based on the fundamental work of Donaldson—proved the existence of continuously-infinite many inequivalent differentiable structures on $\mathbb{R}^4$.

The situation is similar for complex structures, which will be reviewed later in the context of Kähler and Calabi-Yau manifolds. A very recent introduction of the topic of differential and complex structures in physics is presented in [AB07].

A.7. Fiber and vector bundles

Some manifolds can be refined by the notion of fibre bundles, which locally look like a cartesian product of two spaces but may possess a different, nontrivial, global structure. Let E, F and M be three smooth manifolds and $\pi : E \twoheadrightarrow M$ a smooth surjective mapping. Furthermore, require that each point $b \in M$ possesses an open neighborhood $U \subset M$ such that the preimage $\pi^{-1}(U) =: E|_U$ is diffeomorphic to the product space $U \times F$ in the following sense: there exists a mapping $\phi : E|_U \xrightarrow{\approx} U \times F$, called the **local trivialization**, such that the following diagram commutes (i.e. $\mathrm{pr}_1 \circ \phi = \pi$):

$$\begin{tikzcd}
E|_U \arrow[rr, "\substack{\text{local trivialization}\\ \text{(diffeomorphism) } \phi}", "\approx"'] \arrow[dr, "\pi"] & \arrow[d, phantom, "\circlearrowright"] & U \times F \arrow[dl, "\substack{\text{projection}\\ \mathrm{pr}_1:(a,b)\mapsto a}"] \\
& U \arrow[ul, dashed, bend left, "\substack{\text{(local)}\\ \text{section } \sigma}"] &
\end{tikzcd}$$

This is often referred to as **local triviality**, i.e. there exist local trivializations (diffeomorphisms) $\phi : E|_U \xrightarrow{\approx} U \times F$ which establish the local product structure of the bundle. In this context the space E is called the **total space**, M the **base space**, F the **fiber** and π the **bundle projection**. A collection (E, M, F, π) of such objects satisfying the local triviality condition is called a (smooth) **fiber bundle**, and usually denoted as $E \xrightarrow{\pi} M$. A bundle is called **trivial** if it is globally isomorphic to the product bundle $M \times F \xrightarrow{\mathrm{pr}_1} M$.

A (local, smooth) **section** of a fiber bundle is a smooth map $\sigma : U \longrightarrow E$ such that $\pi \circ \sigma = \mathrm{Id}_U$. The space of sections over U is usually denoted as $\Gamma(E|_U)$ or $C^\infty(E|_U)$ in resemblance to the space of smooth functions. If a section is defined on the entire base space $U = M$, it is called a **global section** and denoted $\sigma \in \Gamma(E)$. Fiber bundles and sections can also be understood as a generalization of the notion of function. Since bundles do not in general have globally-defined sections, one of the purposes of the theory of fiber bundles is to measure for the existence of those, which in turn leads to the theory of characteristic

[d] In fact, John W. Milnor was the first to prove the existence of nonstandard differentiable structures on the 7-sphere by considering a nontrivial S^3-bundle over S^4, see [Mil56]. This discovery laid the foundations of differential topology and was honored with the Fields medal six years later. A recent survey on this topic—growing most confusing in dimension four—is found in [Sco05].

[e] Since the differentiable structure on a manifold prescribes which functions are smooth, inequivalent differential structures are quite important in physics. In particular, the central physical property of general coordinate invariance in general relativity depends on the differentiable structure of space-time and there are many inequivalent of those structures. Thus, it does not suffice to know just the topology of space-time (i.e. whether the universe is flat, shaped like a donut, etc.).

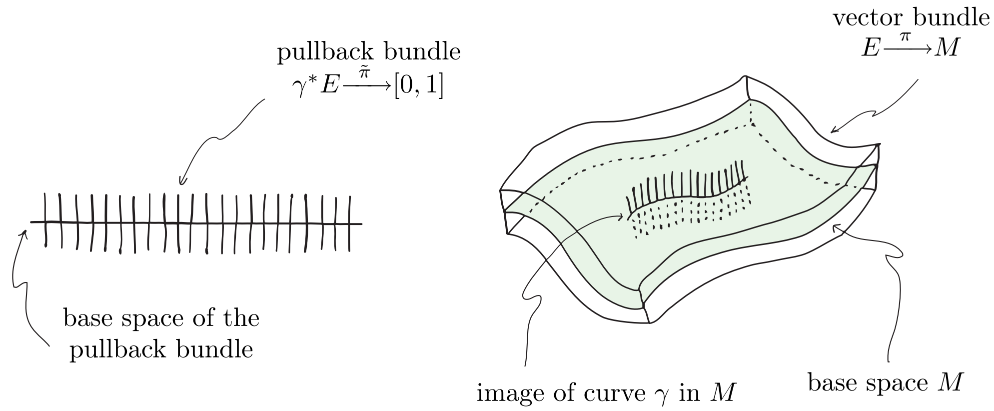


FIGURE A.3. A pullback bundle induced by a smooth curve $\gamma : [0,1] \longrightarrow M$.

classes[f] in algebraic topology. Stiefel-Whitney and Chern classes as well as the Euler class are introduced in chap. 3.

Now, let the fiber F be a real or complex vector space V. If the mapping $v \mapsto \phi^{-1}(b,v)$ for all $v \in F$ and a fixed base point $b \in M$ is then a (real or complex) linear mapping (isomorphism), (E, M, F, π) is called a (real or complex) **vector bundle**. Obviously, vector bundles are just a special case of fiber bundles, where the linearity of a vector space is preserved in the fibers within the total space and in the linearity of the local trivializations. [BJ82, §§3-4] provides a very readable elementary introduction to vector bundle theory.

A natural specialization of vector bundles are **algebra bundles** $\mathcal{A} \xrightarrow{\pi} M$, where every fiber is not just a vector space but in fact an algebra. Obviously, this provides a natural product operation on the vector space of smooth sections $\Gamma(\mathcal{A})$, thus effectively turning $\Gamma(\mathcal{A})$ itself into an algebra. Another important type of fiber bundles are principal G-bundles, where the fiber is a certain type of group. These are introduced in chap. 2.

There is a construction in the theory of fiber bundles to pull a bundle over M back along a smooth mapping to another base space. Let $E \xrightarrow{\pi} N$ be a fiber bundle with abstract fiber F and $f : M \longrightarrow N$ a smooth map. The total space of the **pullback bundle** f^*E is defined by

$$f^*E := \big\{(x,e) \in M \times E : f(x) = \pi(e)\big\},$$

and the new bundle projection is $\tilde{\pi} : f^*E \longrightarrow M$ with $(x,e) \mapsto x$. Thus, $(f^*E, M, F, \tilde{\pi})$ is a fiber bundle over M with fiber F induced by the mapping f. The projection on the second factor $\tilde{f} : f^*E \longrightarrow E$, i.e. $\tilde{f}(x,e) = e$, makes the diagram

$$\begin{tikzcd} f^*E \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\pi}"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

commutative. In particular, any section σ of $E \xrightarrow{\pi} N$ induces a **pullback section** $f^*\sigma$ of $f^*E \xrightarrow{\tilde{\pi}} M$ via $f^*\sigma = \sigma \circ f$. This construction is shown in fig. A.3 and will be used to define the parallel transport in Riemannian geometry in later chapters.

There is an operation reminiscent of the direct sum of vector spaces defined for two vector bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M$ over the same base space. The **Whitney sum** of E and

[f] An extremely well-written account on the subject of characteristic classes is found in [MS74] but requires thorough knowledge of algebraic topology. A modern (and very readable) introduction to algebraic topology is [Hat02]. In [KN69, chp. 12] characteristic classes are introduced from the differential geometric point of view, but the number of results obtained is rather limited.

E' is a vector bundle $E \oplus E' \xrightarrow{\tilde{\pi}} M$ whose fiber over $p \in M$ is the direct sum $E_p \oplus E'_p$ of the respective vector spaces E_p and E'_p. Completely analogous, a **tensor product bundle** $E \otimes E' \xrightarrow{\tilde{\pi}} M$ is constructed. Such tensor product bundles are one of the basic concepts of K-theory, an algebraic theory of vector bundles.

There is a particularly important type of vector bundle constructed as follows: Let $\mathbb{R}\mathcal{P}^n$ be the **real projective *n*-space**, i.e. the quotient space $\mathbb{R}^{n+1}/\mathbb{R}^\times$ which contains the $\mathbb{R}$-linear independent lines trough $0 \in \mathbb{R}^{n+1}$. Define the total space of the bundle as the subset

$$\gamma^n := \Big\{ \big([x], v\big) \in \mathbb{R}\mathcal{P}^n \times \mathbb{R}^{n+1} : v = \lambda x \text{ for } \lambda \in \mathbb{R} \Big\},$$

then the projection $\mathrm{pr}_1 : \big([x], v\big) \mapsto [x] \in \mathbb{R}\mathcal{P}^n$ on the first factor is the bundle projection. The bundle $\gamma^n \xrightarrow{\mathrm{pr}_1} \mathbb{R}\mathcal{P}^n$ is called the **canonical** or **tautological line bundle**. In particular, the total space of γ^1 is just the Möbius strip, constructed as a bundle. Furthermore, one defines the **complex tautological** or **canonical line bundle** $\gamma^n_{\mathbb{C}} \xrightarrow{\mathrm{pr}_1} \mathbb{C}\mathcal{P}^n$ over the complex projective space $\mathbb{C}\mathcal{P}^n := \mathbb{C}^{n+1}/\mathbb{C}^\times$ by replacing $\mathbb{R}$ with $\mathbb{C}$ everywhere in the above definition.

A.8. Tangent spaces and bundles, orientation

Given a differentiable manifold M and a point $p \in M$, the tangent space T_pM is the "closest linear approximation" of the manifold. There are different ways to formulate this in precise mathematical terms, all vividly discussed in [JK06, chp. 2]. Only the "geometrical" and "physical" definition in terms of partial derivatives are presented herein.

With respect to the coordinates $(x_1, \dots, x_n)$ on an open subset of $\mathbb{R}^n$, the partial derivative operators $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ can serve as the basis of a real vector space. Given a point $p \in M$ and a chart (ψ, U) on its neighborhood, the ordinary partial derivatives can be pulled back to the point. The **(physical) tangent space** at p with respect to the chart ψ with local coordinates $(x^1, \dots, x^n)$ is then defined via

$$\mathrm{T}_pM := \operatorname{span}\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p \right\}. \tag{A.1}$$

This particular choice of basis for T_pM, i.e. where the partial derivatives are directly induced from the local coordinates, is sometimes referred to a "holonomic basis". Given another chart ψ' around p with local coordinates $(y_1, \dots, y_n)$ the different partial derivatives are related by the **tensorial transformation law**

$$\frac{\partial}{\partial x^i}\bigg|_p = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}\bigg|_p \quad \text{or} \quad \frac{\partial}{\partial y^i}\bigg|_p = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}\bigg|_p,$$

respectively, which justifies the name "physical" tangent space. Due to the compatibility of the transition functions, this chart-dependent definition of the (physical) tangent space gives rise to the coordinate independent concept of an abstract tangent space.

Another very accessible definition of the tangent space is as follows: Fix a point $p \in M$ and let $\gamma :]-\varepsilon, \varepsilon[\longrightarrow M$ be any differentiable curve with $\gamma(0) = p$. The "velocity vector" $\gamma'(0)$ then obviously provides a linear approximation of M in a single direction. Define

$$\mathcal{C}_pM := \big\{ \text{differentiable curves } \gamma :]-\varepsilon, \varepsilon[\longrightarrow M \text{ with } \gamma(0) = p \big\},$$

then two curves $\alpha, \beta \in \mathcal{C}_pM$ are called equivalent if for any chart (ψ, U) around $p \in U$ the equality

$$\frac{d}{dt}(\psi \circ \alpha)\bigg|_{t=0} = \frac{d}{dt}(\psi \circ \beta)\bigg|_{t=0}$$

is satisfied, i.e. curves having the same "velocity vector" at p are identified under this equivalence relation "$\sim$". The set of equivalence classes

$$\mathrm{T}_pM := \mathcal{C}_pM / \sim \tag{A.2}$$

is called the **(geometric) tangent space** attached to p. The equivalence of (A.1) and (A.2) is easily proved by usage of the chain rule, see [JK06, §2.3].

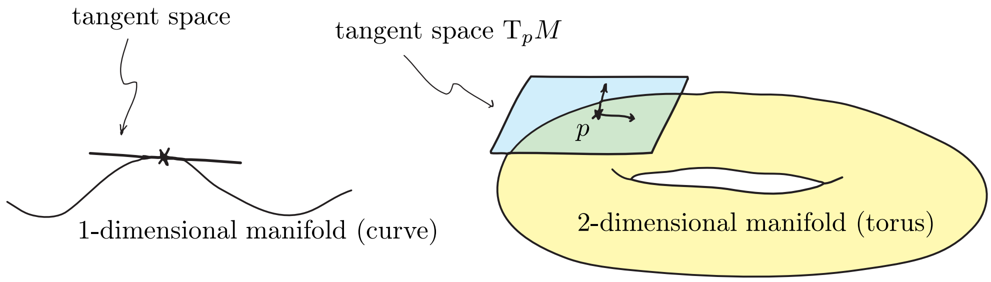


FIGURE A.4. Geometric depiction of 1- and 2-dimensional tangent spaces.

For a complex n-dimensional manifold M one has to introduce the concept of a **complex tangent space** $\mathrm{T}_p^{\mathbb{C}}M$. Regarded as a real vector space, $\mathrm{T}_p^{\mathbb{C}}M$ is the same as the real tangent space $\mathrm{T}_p(M_{\mathbb{R}})$ of the underlying real $2n$-dimensional manifold $M_{\mathbb{R}}$. Using the identification $z = x + \mathrm{i}y$ of complex and real coordinates, the complex tangent space can be written with respect to the local complex coordinates $(z_1, \ldots, z_n)$ by a linear coordinate transformation

$$\begin{aligned}\mathrm{T}_p^{\mathbb{C}}M :=& \operatorname{span}\left\{\left(\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^{n+1}}\right)\bigg|_p, \ldots, \left(\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^{n+1}}\right)\bigg|_p, \ldots\right\}\\ =& \operatorname{span}\left\{\frac{\partial}{\partial z^1}\bigg|_p, \ldots, \frac{\partial}{\partial z^n}\bigg|_p, \frac{\partial}{\partial \bar{z}^1}\bigg|_p, \ldots, \frac{\partial}{\partial \bar{z}^n}\bigg|_p\right\}.\end{aligned}$$

Thus, the complex tangent space has complex dimension n and real dimension $2n$ as obvious from the above definition. One can now pull apart the complex tangent space in the real vector spaces $\mathrm{T}_p^{\mathbb{C}}M = \mathrm{T}_p^{(1,0)}M \oplus \mathrm{T}_p^{(0,1)}M$ where

$$\mathrm{T}_p^{(1,0)}M := \operatorname{span}\left\{\frac{\partial}{\partial z^1}\bigg|_p, \ldots, \frac{\partial}{\partial z^n}\bigg|_p\right\}, \qquad \text{(holomorphic derivatives)}$$

$$\mathrm{T}_p^{(0,1)}M := \operatorname{span}\left\{\frac{\partial}{\partial \bar{z}^1}\bigg|_p, \ldots, \frac{\partial}{\partial \bar{z}^n}\bigg|_p\right\} \qquad \text{(antiholomorphic derivatives)}$$

are the **holomorphic tangent space** and **antiholomorphic tangent space**.

Using the idea of a vector bundle, one places the tangent spaces at all points of the manifold M as a disjoint union into a total space, i.e.

$$\mathrm{T}M := \coprod_{p\in M} \mathrm{T}_pM$$

specifies an appropriate topology (for details see [Hus98, §18.3]) and uses the canonical projection $\pi : \mathrm{T}M \longrightarrow M$ onto the base point where each respective tangent space is attached to. This definition yields the **tangent bundle** $\mathrm{T}M \xrightarrow{\pi} M$. The **complex tangent bundle** $\mathrm{T}M^{\mathbb{C}}$ is defined analogous, and the (anti-)holomorphic splitting $\mathrm{T}M^{\mathbb{C}} = \mathrm{T}M^{(1,0)} \oplus \mathrm{T}M^{(0,1)}$ also holds for bundles as well.

Suppose M and N are differentiable manifolds and $f : M \longrightarrow N$ is a differentiable mapping between them. Then there exists a natural linear mapping

$$\mathrm{T}f_p : \mathrm{T}_pM \longrightarrow \mathrm{T}_{f(p)}N$$

between the respective tangent spaces, called either the **tangent mapping** or **push-forward**. Using the geometric definition of the tangent space, where $\gamma'(0)$ is any tangent vector,

$$\mathrm{T}f_p\big(\gamma'(0)\big) = (f \circ \gamma)'(0)$$

gives the mapping in explicit form.[g] The tangent mapping is nothing else than the natural generalization of derivatives of smooth functions $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ to smooth mappings of manifolds $M \longrightarrow N$. Given three manifolds L, M, N and diffentiable mappings $f : L \longrightarrow M$ and $g : M \longrightarrow N$, the **chain rule** of differentiation reads

$$\begin{tikzcd} \mathrm{T}_p L \arrow[r, "\mathrm{T}f_p"] \arrow[rr, bend right, "\mathrm{T}(g\circ f)_p"'] & \mathrm{T}_{f(p)} M \arrow[r, "\mathrm{T}g_{f(p)}"] & \mathrm{T}_{g\circ f(p)} N \end{tikzcd} \qquad \rightsquigarrow \qquad \mathrm{T}(g \circ f)_p = \mathrm{T}g_{f(p)} \circ \mathrm{T}f_p$$

in this coordinate independent notation for any point $p \in L$. Of course, with respect to local coordinates, this gives the chain rule in terms of partial derivatives.

A.9. The classical groups

Let V be a finite dimensional vector space over the field $F = \mathbb{R}$ or $\mathbb{C}$. The group of automorphisms $\mathrm{Aut}(V) = \{A \in \mathrm{End}(V) : \det A \neq 0\}$ is a open subset of the finite-dimensional vector space $\mathrm{End}(V)$. These groups are called the **general linear groups**

$$\mathrm{GL}_n(\mathbb{R}) := \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^n), \qquad \text{(real general linear group)}$$
$$\mathrm{GL}_n(\mathbb{C}) := \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^n). \qquad \text{(complex general linear group)}$$

After a choice of basis for V, linear maps $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ are described by $n \times m$-matrices. In particular, $\mathrm{GL}_n(\mathbb{R})$ is canonically isomorphic to the group of invertible quadratic real $n \times n$-matrices. Thus, $\mathrm{GL}_n(\mathbb{R})$ and all its classical subgroups (there are "unusual" subgroups in the context of spin geometry, see later chapters) are called **matrix groups**. In the context of vector space orientation there are two important subgroups

$$\mathrm{GL}_n^+(\mathbb{R}) := \{A \in \mathrm{GL}_n(\mathbb{R}) : \det A > 0\} \qquad \text{(orientation-preserving automorphisms)}$$
$$\mathrm{GL}_n^-(\mathbb{R}) := \{A \in \mathrm{GL}_n(\mathbb{R}) : \det A < 0\} \qquad \text{(orientation-changing automorphisms)}$$

for the real case. Note that such a splitting of the group is not possible in the complex case as $\det \mathrm{GL}_n(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ is a connected set, whereas $\det \mathrm{GL}_n(\mathbb{R}) = \mathbb{R} \setminus \{0\}$ consists of two connection components. Because of this, complex manifolds—or rather their underlying real manifolds—are always orientable in contrast to the real case. One may also restrict to orientation-preserving automorphism groups with fixed determinant, i.e.

$$\mathrm{SL}_n(\mathbb{R}) := \{A \in \mathrm{GL}_n(\mathbb{R}) : \det A = 1\}, \qquad \text{(real special linear group)}$$
$$\mathrm{SL}_n(\mathbb{C}) := \{A \in \mathrm{GL}_n(\mathbb{C}) : \det A = 1\}, \qquad \text{(complex special linear groups)}$$

which are called the **special linear groups**. Let $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ denote the multiplicative subgroups of the respective fields, then the **projective groups** are defined via

$$\mathrm{PGL}_n(\mathbb{R}) := \mathrm{GL}_n(\mathbb{R})/\mathbb{R}^\times, \qquad \text{(real projective linear group)}$$
$$\mathrm{PGL}_n(\mathbb{C}) := \mathrm{GL}_n(\mathbb{C})/\mathbb{C}^\times. \qquad \text{(complex projective linear group)}$$

The projective groups are transformation groups of the projective spaces $\mathbb{R}\mathcal{P}^n$ and $\mathbb{C}\mathcal{P}^n$.

Given any additional structure on the vector space, one can define further invariance groups. The **orthogonal groups**

$$\mathrm{O}(n) := \{A \in \mathrm{GL}_n(\mathbb{R}) : A^{\mathsf{t}} A = \mathbb{1}\} \qquad \text{(invariance group of } \langle .,. \rangle_{\mathbb{R}^n})$$

can be understood as the standard inner product preserving invariance group of $\mathbb{R}^n$. Likewise, for complex vector spaces the **unitary groups**

$$\mathrm{U}(n) := \{A \in \mathrm{GL}_n(\mathbb{C}) : A^\dagger A = \mathbb{1}\} \qquad \text{(invariance group of } \langle .,. \rangle_{\mathbb{C}^n})$$

[g] The reader should be aware of the various notations used in the literature: $\mathrm{d}f_p$, $\mathrm{D}f_p$, f_* and $f'(p)$ all refer to the tangent mapping $\mathrm{T}f_p$ as defined above.

preserve the standard unitary inner product on $\mathbb{C}^n$, where $A^\dagger = \bar{A}^{\mathrm{t}}$ denotes complex conjugation and transposition of the matrix elements. The orthogonal subgroup $\mathrm{O}(n) \subset \mathrm{GL}_n(\mathbb{R})$ splits in two connection components by the values ± 1 of the determinant. The orientation-preserving one of these two is called the **special orthogonal group**

$$\mathrm{SO}(n) := \mathrm{O}(n) \cap \mathrm{GL}_n^+(\mathbb{R}) = \{A \in \mathrm{O}(n) : \det A = 1\}.$$

Thus, $\mathrm{SO}(n)$-transformations preserve both the orientation and the standard inner product on $\mathbb{R}^n$, i.e. they are **rotations**. Analogous to this, the **special unitary group** is defined by

$$\mathrm{SU}(n) := \{A \in \mathrm{U}(n) : \det A = 1\},$$

which preserves the standard unitary product on $\mathbb{C}^n$ and the determinant. All those groups are compact subgroups of $\mathrm{GL}_n(F)$, i.e. they are closed and bounded subsets in the finite-dimensional vector space $\mathrm{End}(V)$.

A.10. Representation of groups

Let G be a group and V a vector space over the field F. A **(linear) representation**[h] of G on V over F is a group homomorphism $\rho : G \longrightarrow \mathrm{Aut}(V) = \mathrm{GL}(V)$. The vector space V is called the **representation space** and the dimension of V is the **dimension** of the representation. The representation is called **faithful** if the representation homomorphism ρ is injective. The set of all representations of a group is denoted $\mathrm{Rep}(G)$.

A representation assigns to every group element a linear operator acting on the representation vector space, which after choosing a basis for the representation space corresponds to an invertible square matrix due to $\mathrm{GL}(V) \cong \mathrm{GL}_n(F) \subset \mathrm{Mat}(n \times n; F)$. This specific representation $\rho : G \longrightarrow \mathrm{GL}_n(F)$ is called the n-dimensional **matrix representation** of a group. Abstract representations (without a choice of basis) and matrix representations are often mixed up in the notation and will not be distinguished.

Two representations ρ, ρ' of the same group are **equivalent** if there exists an automorphism $S \in \mathrm{GL}(V)$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\rho_g"] \arrow[d, "S"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "\cong"', "S"] \\ V \arrow[r, "\rho'_g"'] & V \end{tikzcd} \qquad \rightsquigarrow \qquad \rho'_g = S \rho_g S^{-1}$$

For matrix representations this is to be understood as a matrix equation, i.e. the matrices of the respective representations are related by a change of basis in the representation space.

An **unitary representation** of a group G is a (linear) representation ρ of G on a complex Hilbert space $\mathcal{H}$, such that ρ_g is an unitary operator for every $g \in G$. For short, one has the group homomorphism $\rho : G \longrightarrow \mathrm{U}(\mathcal{H})$. After choosing a basis for the Hilbert space, the canonical isomorphism to the set of unitary matrices gives rise to a (unitary) matrix representation just like in the general case. An elementary result from group theory provides, that every representations of a compact group can be made unitary, see [BtD85, thm. II.1.7]. This makes the representation theory of the non-compact Lorentz and Poincaré group quite problematic.

Let $\rho_1 : G \longrightarrow V_1$ and $\rho_2 : G \longrightarrow V_2$ be two representations of the same group G. The **sum representation** $\rho_1 \oplus \rho_2$ acting on the representation space $V_1 \oplus V_2$ is defined (in matrix notation) by

$$\rho_1 \oplus \rho_2(g) := \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right).$$

[h] In general, a representation is just a homomorphism $\rho : G \longrightarrow \mathrm{Aut}(W)$ into the automorphism group of an arbitrary set W. Such non-linear representations are usually not considered in physics and even in pure mathematics are of rather limited interest due to the lack of any additional structure.

This representation has dimension $n_1 + n_2$. Likewise, a **tensor product representation** $\rho_1 \otimes \rho_2$ acting on $V_1 \otimes V_2$ is defined component-wise as

$$\big[\rho_1 \otimes \rho_2\big]_{im,jn} = \big[\rho_1(g)\big]_{ij}\big[\rho_2(g)\big]_{mn}$$

and has the dimension $n_1 n_2$. In the same way other vector space constructions extend naturally to representation theory, e.g. the symmetric and antisymmetric products introduced in the next section.

A representation ρ of the group G is **reducible** if it is equivalent to a sum representation, i.e. the representation space and the representation can be split up into ρ_1 acting on V_1 and ρ_2 acting on V_2, respectively. An **irreducible** representation is one not reducible, and all representations of a compact group are in fact direct sums of irreducible representations. Thus, the irreducible representations are of special importance, as they are the building blocks of any further representations. In general a tensor product representation is reducible and its decomposition in a sum of irreducible representations can be systematically solved using Clebsch-Gordan coefficients, see [BtD85, §VI.3]. This is used for the addition of angular momentums in quantum mechanics, see [Sak85, §3.7] or [AW66, §4.4] for a physical and [Hal03, app. D] for a mathematical treatment. [SU01, §7.8, §8.3] provides a treatise on this subject particularly useful for physicists.

A.11. Exterior algebra

Differential forms are an important construction to express geometrical ideas independent from a choice of local coordinates. Furthermore, differential k-forms generalize the concept of charges and fields in physics. A k-form defined on the the space-time $\mathcal{M}$ is an abstract quantity which only takes an actual, physical relevant value if it is integrated over an k-dimensional submanifold of $\mathcal{M}$. This is analogous to the abstract electric field $\vec{E}$ of a point charge q which "permeates" the entire space but whose actual physical effect of Coulomb force is only measurable with respect to certain test charges. In string theory the Kalb-Ramond 2-form field B is the most prominent example of a n-form field.

Let V be a finite-dimensional real vector space, then one can consider the tensor product $V \otimes V$ with elements denoted by $v \otimes w$. Naturally, one might consider only symmetric or antisymmetric products, i.e. elements satisfying either $v \otimes w = w \otimes v$ or $v \otimes w = -w \otimes v$, respectively. The ***k*-th exterior power** $\Lambda^k V$ denotes the subspace of antisymmetric products in the k-fold tensor product $V^{\otimes k}$. Similarly, one defines the ***k*-th symmetric power** $\mathrm{S}^k V$, such that the splitting $V^{\otimes k} = \Lambda^k V \oplus \mathrm{S}^k V$ holds for $k > 1$. Somewhat inconsistently, one usually defines both $\Lambda^1 V$ and $\mathrm{S}^1 V$ to be the vector space V itself. Given a n-dimensional vector space, $\Lambda^k V$ has dimension $\binom{n}{k}$ for $k \leq n$, as easily seen from combinatorial considerations, and $\Lambda^k V = 0$ for all $k > n$.

The direct sum of all antisymmetric powers together with an antisymmetric product constitutes a graded algebra, called the **exterior algebra** or **Grassmann algebra** $\Lambda^\bullet V := \bigoplus_{k=0}^\infty \Lambda^k V$. For $\omega \in \Lambda^k V$ and $\eta \in \Lambda^l V$ the **wedge product** is defined via

$$\begin{aligned} \wedge : \Lambda^k V \otimes \Lambda^l V &\longrightarrow \Lambda^{k+l} V \\ \omega \otimes \eta &\mapsto \omega \wedge \eta := \mathrm{Alt}(\omega \otimes \eta) \end{aligned}$$

to be the antisymmetric part of the tensor product, i.e. $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$, which gives an element of $\Lambda^{k+l} V$. For direct sums of vector spaces the k-th exterior power reads

$$\Lambda^k (V \oplus W) = \bigoplus_{i+j=k} \Lambda^i V \otimes \Lambda^j W. \tag{A.3}$$

Further algebraic properties of the wedge product are found in most algebra textbooks, e.g. [Lan93, chp. 19].

A.12. Differential forms

In general, for any vector space V over the field F one can define the **dual vector space** V^* of F-linear maps $V \longrightarrow F$, which itself is a vector space over the field F again. Given any basis $(e_1, \dots, e_n)$ for V, the associated dual basis for V^* is $(e^1, \dots, e^n)$ and determined by the relations $e^i(e_j) = \delta^i_j$.

Naturally, the dual tangent space T^*_pM, called the **cotangent space**, is of special interest. Aside from the canonical definition as a dual vector space associated to T_pM, it can be understood in a more geometrical way: Consider a smooth real-valued function $f \in C^\infty(M)$ defined on the manifold, then consider the tangent mapping $\mathrm{T}f_p : \mathrm{T}_pM \longrightarrow \mathrm{T}_{f(p)}\mathbb{R}$ at a point $p \in M$. Due to the canonical isomorphism $\mathrm{T}_{f(p)}\mathbb{R} \cong \mathbb{R}$ this is a linear, real-valued mapping $\mathrm{T}_pM \longrightarrow \mathbb{R}$, i.e. $\mathrm{T}f_p \in \mathrm{T}^*_pM$. Thus, $\mathrm{T}f_p(v)$ gives the derivative at p of the function f in the direction $v \in \mathrm{T}_pM$. This point-wise construction canonically extends by usage of the tangent and cotangent bundle concept, i.e. there is a mapping $\mathrm{T}f : \mathrm{T}M \longrightarrow \mathbb{R}$ which gives the directional derivative of X at each point p after inserting a proper direction vector $v_p \in \mathrm{T}_pM$, which is called the **differential** of f and (in this context) is usually denoted $\mathrm{d}f := \mathrm{T}f$.

For the tangent space T_pM with respect to local coordinates $(x_1, \dots, x_n)$ on $U \subset M$, a convenient choice of dual basis elements are $\mathrm{d}x^1|_p, \dots, \mathrm{d}x^n|_p$, where each $\mathrm{d}x^i|_p : \mathrm{T}_pM \longrightarrow \mathbb{R}$ is a $\mathbb{R}$-linear mapping such that the orthogonality relation

$$\mathrm{d}x^i|_p \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) = \mathrm{d}x^i|_p \Big(\partial_j|_p \Big) = \delta^i_j$$

holds, thus $\mathrm{T}^*_pM = \operatorname{span}\left\{ \mathrm{d}x^1|_p, \dots, \mathrm{d}x^n|_p \right\}$. Placing all cotangent spaces into the **cotangent bundle** $\mathrm{T}^*M \xrightarrow{\pi} M$ in the same way as before, a natural local basis of sections for $\Gamma(\mathrm{T}^*M|_U)$ is given by $\mathrm{d}x^1, \dots, \mathrm{d}x^n$. These sections $\mathrm{d}x^i$ are called (local) **1-differential forms** or just **1-forms** for short. Using the chain rule of differentiation, the differential $\mathrm{d}f$ of any smooth function $f \in C^\infty(M)$ can be written as

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \, \mathrm{d}x^i.$$

One should acknowledge that the left-hand side of this equality is coordinate-independent by construction, whereas the right-hand side is just a local representation in terms of local coordinates as used for actual calculations.

The exterior algebra is usually used in connection with multilinear maps, specifically the cotangent space T^*_pM. The exterior power $\Lambda^k\mathrm{T}^*_pM$ is the space of antisymmetric multilinear mappings $\mathrm{T}_pM \times \cdots \times \mathrm{T}_pM \longrightarrow \mathbb{R}$, and the wedge product can be expressed explicitly as

$$\omega \wedge \eta(v_1, \dots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \operatorname{sign}(\sigma) \omega\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big) \eta\big(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}\big)$$

for $\omega \in \Lambda^k\mathrm{T}^*_pM$ and $\eta \in \Lambda^l\mathrm{T}^*_pM$, where the sum ranges over all permutations of the $k + l$ tangent vectors. Again, this definition extends to bundles, and the sections of the k-th exterior power of the cotangent bundle (which itself is a bundle $\Lambda^k\mathrm{T}^*M \xrightarrow{\pi} M$) are called ***k*-differential forms** or ***k*-forms** on M denoted as $\Omega^k_M := \Gamma(\Lambda^k\mathrm{T}^*M)$. A very readable reference for both the exterior algebra and differential forms is [GP74, §§4.1-3].

All of these definitions are completely analogous for complex manifolds and complex tangent spaces under the usage of the (anti-)holomorphic splitting. Let N be a complex n-dimensional manifold. The **holomorphic *i*-forms** are contained in $\Omega^{i,0}_N := \Gamma(\Lambda^i\mathrm{T}^{(1,0)*}N)$, whereas the **antiholomorphic *j*-forms** are in $\Omega^{0,j}_N := \Gamma(\Lambda^j\mathrm{T}^{(0,1)*}N)$. For higher exterior powers with $k > 1$, using (A.3), the space of **complex *k*-forms** $\Omega^k_{N;\mathbb{C}} := \Gamma(\Lambda^k\mathrm{T}^{*\mathbb{C}}M)$ splits like

$$\Omega^k_{N;\mathbb{C}} = \bigoplus_{i+j=k} \Omega^{i,0}_N \otimes \Omega^{0,j}_N = \bigoplus_{i+j=k} \Omega^{i,j}_N,$$

as one would expect from the splitting of the corresponding complex tangent space. The elements of $\Omega_N^{i,j} := \Omega_N^{i,0} \otimes \Omega_N^{0,j}$ are called **(i,j)-forms**. Further information about complex differential forms can be found in [Huy05, §2.6].

Any real k-form ω can locally be expressed as a wedge product of (local) 1-forms, i.e. with respect to local coordinates $(x_1, \ldots, x_n)$ it follows

$$\begin{aligned}\omega|_U &= \sum_{a,b,\ldots,d} \frac{1}{k!}\omega_{ab\ldots d} \underbrace{\mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \cdots \wedge \mathrm{d}x^d}_{k\text{-fold wedge product}} \\ &= \sum_{a<b<\ldots<d} \omega_{ab\ldots d}\, \mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \cdots \wedge \mathrm{d}x^d,\end{aligned}$$

where $\omega_{ab\ldots d}$ are the total antisymmetric components physicists use for calculations.[i] Likewise, (i,j)-forms can be locally expressed by

$$\eta|_U = \sum_{\substack{a,\ldots,d,\\ \bar{a},\ldots,\bar{e}}} \frac{1}{i!j!}\eta_{ab\ldots d\bar{a}\bar{b}\ldots\bar{e}} \underbrace{\mathrm{d}z^a \wedge \cdots \wedge \mathrm{d}z^d}_{i \text{ times}} \wedge \underbrace{\mathrm{d}\bar{z}^{\bar{a}} \wedge \cdots \wedge \mathrm{d}\bar{z}^{\bar{e}}}_{j \text{ times}}.$$

Together with the point-wise well-defined wedge product, this gives rise to the graded algebra of differential forms $\Omega_M^\bullet$ on any differentiable manifold M or complex differential forms $\Omega_{N,\mathbb{C}}^\bullet$ on any complex manifold N, respectively.

A.13. Exterior differentiation

Besides the wedge product there is another natural operation on differentiable manifolds, called the **exterior differentiation** $\mathrm{d} : \Omega_M^p \longrightarrow \Omega_M^{p+1}$ of forms. With respect to local coordinates $(x_1, \ldots, x_n)$ on $U \subset M$, for any $\omega \in \Omega_M^p$ this operations is defined by

$$\omega \mapsto \mathrm{d}\omega = \sum_{a,\ldots,d,k} \frac{\partial \omega_{a\ldots d}}{\partial x^k}\, \mathrm{d}x^k \wedge \mathrm{d}x^a \wedge \cdots \wedge \mathrm{d}x^d.$$

Note that the added 1-form $\mathrm{d}x^k$ is put in front of the other differential forms to avoid any unnecessary sign ambiguities. Furthermore, one can prove that the exterior derivative commutes with pull-backs, i.e. $\mathrm{d}(f^*\omega) = f^*(\mathrm{d}\omega)$.

For complex manifolds the situation is a little more involved due to the complex nature: One has two distinct differentiations, the **holomorphic** and **antiholomorphic exterior differentiation**

$$\begin{aligned}\partial : \Omega_N^{r,s} &\longrightarrow \Omega_N^{r+1,s} \\ \eta \mapsto \partial\eta &= \sum_{\substack{a,\ldots,d,k,\\ \bar{a},\ldots,\bar{d}}} \frac{\partial \eta_{a\ldots d\bar{a}\ldots\bar{d}}}{\partial z_k}\, \mathrm{d}z^k \wedge \mathrm{d}z^a \wedge \cdots \wedge \mathrm{d}z^d \wedge \mathrm{d}\bar{z}^{\bar{a}} \wedge \cdots \wedge \mathrm{d}\bar{z}^{\bar{d}} \\ \bar{\partial} : \Omega_N^{r,s} &\longrightarrow \Omega_N^{r,s+1} \\ \eta \mapsto \bar{\partial}\eta &= \sum_{\substack{a,\ldots,d,\\ \bar{a},\ldots,\bar{d},\bar{k}}} \frac{\partial \eta_{a\ldots d\bar{a}\ldots\bar{d}}}{\partial \bar{z}_{\bar{k}}}\, \mathrm{d}\bar{z}^{\bar{k}} \wedge \mathrm{d}z^a \wedge \cdots \wedge \mathrm{d}z^d \wedge \mathrm{d}\bar{z}^{\bar{a}} \wedge \cdots \wedge \mathrm{d}\bar{z}^{\bar{d}}\end{aligned}$$

which can be interpreted as a refinement of the exterior differentiation, as $\mathrm{d}\eta^{r,s} = \partial\eta^{r,s} + \bar{\partial}\eta^{r,s}$ holds for any $\eta \in \Omega_{N;\mathbb{C}}^{r,s}$ when regarded as a real $(r+s)$-form, see [Huy05].

[i] There are different conventions regarding the combinatorial factor in the local representation of a differential form used in the literature. The choice here allows for direct identification with physical quantities.

A.14. Bundle-valued differential forms

Another important aspect is to consider special target spaces for differential forms on M, in particular a vector bundle $E \xrightarrow{\pi} M$. In general, the tensor product

$$\Omega^k_M(E) := \Gamma(\Lambda^k \mathrm{T}^* M \otimes E) = \Omega^k_M \otimes \Gamma(E)$$

denotes the space of ***E*-valued *k*-forms**. Furthermore, any vector space V can be seen as a trivial product bundle $M \times V \xrightarrow{\mathrm{pr}_1} M$, which gives rise to V-valued k-forms $\Omega^k_M(V)$.

It is important to realize that the collection of all E-valued k-forms cannot constitute an algebra like ordinary real-valued differential forms as in $\Omega^\bullet_M$ due to lack of a natural generalization of the wedge product. However, there exists a well-defined product between bundle-valued and real-valued forms: Any E-valued k-form $\omega \in \Omega^k_M(E)$ can be split-represented as $\omega = \tilde{\omega} \otimes \omega_E$, where $\tilde{\omega} \in \Omega^k_M$ is a real-valued k-form and $\omega_E \in \Gamma(E)$ a section of the vector bundle. Using this splitting, the natural wedge product extension takes the form

$$\begin{aligned} \wedge : \Omega^k_M(E) \otimes \Omega^l_M &\longrightarrow \Omega^{k+l}_M(E) \\ \omega \otimes \tilde{\eta} &\mapsto \omega \wedge \tilde{\eta} := (\tilde{\omega} \wedge \tilde{\eta}) \otimes \omega_E. \end{aligned}$$

Naturally, algebra-bundle-valued differential forms do not suffer from the same algebraic problems. Let $\mathcal{A} \xrightarrow{\pi} M$ be an algebra bundle over M, then this defines **$\mathcal{A}$-valued *k*-forms**, where the wedge product is generalized via

$$\begin{aligned} \wedge : \Omega^k_M(\mathcal{A}) \otimes \Omega^l_M(\mathcal{A}) &\longrightarrow \Omega^{k+l}_M(\mathcal{A}) \\ \omega \otimes \eta &\mapsto \omega \wedge \eta := (\tilde{\omega} \wedge \tilde{\eta}) \otimes \omega_{\mathcal{A}} \eta_{\mathcal{A}}, \end{aligned}$$

where $\omega_{\mathcal{A}}\eta_{\mathcal{A}}$ refers to the product operation induced on $\Gamma(\mathcal{A})$. Note that the (anti-)symmetry of the algebra product affects the ordinary antisymmetry $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$ of the forms. In particular, the Lie-algebra-valued differential forms will become important in the gauge theory chapter.

A.15. Vector fields, determinants and orientation

Sections of the tangent bundle are usually called **vector fields**,[j] and $\mathfrak{X}(M) := \Gamma(\mathrm{T}M)$ denotes the space of vector fields, which is in fact an infinite-dimensional Lie algebra. This will be elaborated further in the next chapter. With respect to local coordinates $(x_1, \dots, x_n)$ on $U \subset M$ any vector fields $V, W \in \mathfrak{X}(M)$ may be written as

$$V = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i}, \qquad W = \sum_{j=1}^n W^i \frac{\partial}{\partial x^j}$$

with local component functions $V^i, W^i : U \longrightarrow \mathbb{R}$. Let $f : U \longrightarrow \mathbb{R}$ be a real-valued function, then $Vf : U \longrightarrow \mathbb{R}$ is to be understood as the directional derivative of the function. Considering the formal commutator

$$\begin{aligned} (VW - WV)f &= \sum_{i,j=1}^n V^i \frac{\partial}{\partial x^i} \left(W^j \frac{\partial}{\partial x^j} f \right) - \sum_{i,j=1}^n W^j \frac{\partial}{\partial x^j} \left(V^i \frac{\partial}{\partial x^i} f \right) \\ &= \sum_{j=1}^n \left[\sum_{i=1}^n \left(V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i} \right) \right] \frac{\partial}{\partial x^j} f \end{aligned}$$

proves that (at least locally) $VW - WV$ yields a new vector field on U. By further considerations (see [Jos95, §1.6]) this result can be extended globally, thus $[V, W] := VW - WV$ defines an antisymmetric product operation

$$\begin{aligned} [.,.] : \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ (V, W) &\mapsto [V, W] = VW - WV \end{aligned}$$

[j] This definition of vector fields is not to be confused with the "vector fields" or "vector potentials" used in physics. Such objects will be discussed in the context of gauge field theory in the later chapters.

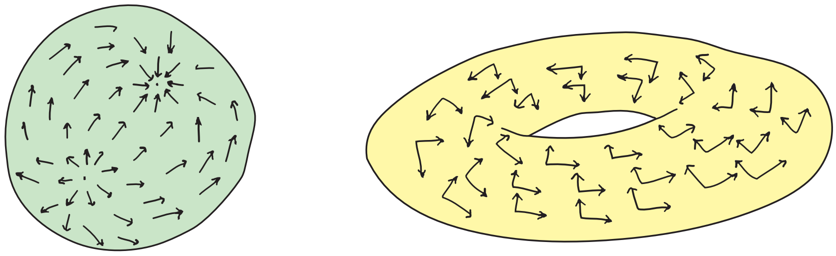


FIGURE A.5. A vector field with two zeros on a 2-sphere and two vector fields on the 2-torus that give a basis for $\Gamma(\mathrm{T}M)$.

on the set of smooth vector fields on M, called the **Lie bracket of vector fields**. The Lie bracket will be introduced from much more general viewpoint in the next sections.

Since the 1-dimensional space $\Lambda^n(\mathbb{R}^n)^*$ consists of multilinear mappings in n antisymmetric parameters, any antisymmetric multilinear mapping $\omega \in \Lambda^n(\mathbb{R}^n)^*$ can be written as

$$\omega(v_1, \dots, v_n) = C_\omega \det(v_1, \dots, v_n)$$

for a constant C_ω. Of course, this generalizes to exterior powers of bundles where vector fields are inserted instead of vectors. Given an n-dimensional manifold M, the **determinant line bundle** $K_M := \Lambda^n \mathrm{T}^* M \xrightarrow{\pi} M$ is the highest non-trivial power of the cotangent bundle.[k]

The determinant line bundle can be used to define the **orientation of a manifold** M. A nowhere vanishing global section $\mathrm{dvol} \in \Gamma(K_M) = \Gamma(\Lambda^n \mathrm{T}^* M) = \Omega^n_M$ is called a **volume form** and provides the notions “left-handedness” and “right-handedness” for each tangent space $\mathrm{T}_p M$ that varies smoothly with the base point p. Topological obstructions may prevent the possibility of such a choice, e.g. the Möbius strip or the Klein bottle are non-orientable.

A.16. Pull-backs and push-forwards

Tangent and cotangent bundles, as well as vector fields and forms, have important categorical properties in the context of differentiable mappings. A proper understanding of their behavior when pulled back or pushed forward along a mapping of the base space makes the underlying concepts much clearer and is needed in the main chapters. Let $V \in \mathfrak{X}(M)$ be a vector field on M, then $f_* V$ yields a vector field on N, i.e. the vector field is—just as the name suggests—pushed forward along the direction of the mapping:

$$\begin{tikzcd}
\mathrm{T}M \arrow[r, squiggly, "\text{push-forward } f_*"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{T}N \arrow[d, "\pi'"] \\
M \arrow[r, "f"'] \arrow[u, dashed, bend left=60, "\substack{\text{original vector}\\ \text{field } V}"] & N \arrow[u, dashed, bend right=60, "\substack{\text{pushed vector}\\ \text{field } f_*V}"']
\end{tikzcd}$$

The chain-rule of differentiation provides $(g \circ f)_* = g_* \circ f_*$, which is often called the “functoriality of the push-forward”.

Considering cotangent spaces and bundles, everything is turned backwards. Dual to the push-forward, there is the **pull-back** $f^* : T^*N \longrightarrow T^*M$ defined by

$$(f^*\alpha)(V) = \alpha(f_* V)$$

for a 1-form α and any vector field $V \in \mathfrak{X}(M)$. In this case, the functoriality is of reversed order $(g \circ f)^* = f^* \circ g^*$ and often referred to as “cofunctoriality of the pull-back”. In essence,

[k]Note that this is also called the “canonical bundle associated to M” in some of the literature—however, this terminology is not used herein in order to avoid confusion with the tautological line bundle γ^n previously defined in sec. A.7.

a 1-form is pulled back opposite to the direction of the mapping:

$$\begin{tikzcd} \mathrm{T}^*M \arrow[d, "\pi"'] \arrow[drr, phantom, "\circlearrowleft"] & & \mathrm{T}^*N \arrow[ll, squiggly, "\text{pull-back } f^*"'] \arrow[d, "\pi'"] \\ M \arrow[u, dashed, bend left=60, "\substack{\text{pulled}\\ \text{1-form } f^*\alpha}"] \arrow[rr, "f"'] & & N \arrow[u, dashed, bend right=60, "\substack{\text{original}\\ \text{1-form } \alpha}"'] \end{tikzcd}$$

In the context of the exterior algebra, the pull-back canonically extends to higher differential forms via $f^*\beta(V_1, \ldots, V_k) = \beta(f_*V_1, \ldots, f_*V_k)$ for any $\beta \in \Omega^k_N$. A general introduction to category theory by one of its founders is found in the classic text [Mac98].

A.17. Lie groups and algebras

The categorical aspects of vector fields discussed in the last section, allow to introduce a certain type of group, which is of great importance both in physics and mathematics. A (real) **Lie group** is a group whose topological set is also a n-dimensional smooth manifold. Essentially, the elements can be labeled by a set of continuous real parameters ξ^a, $a = 1, \ldots, n$, with the composition law depending smoothly on the parameters, i.e. there is a smooth function f satisfying $g(\xi) * g(\xi') = g\big(f(\xi, \xi')\big)$. Usually those coordinates are chosen in a way such that $\xi = 0$ represents the unit element. Complex Lie groups are defined analogous. In physics **compact Lie groups** are of special importance, which means the group manifold is compact as a topological space. In particular, any compact group has unitary representations, as it was already mentioned in sec. A.10.

A (real) **Lie algebra** is a vector space $\mathfrak{g}$ equipped with an antisymmetric bilinear mapping $[.,.]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, called the **Lie bracket**, satisfying the **Jacobi identity**

$$\big[X, [Y, Z]_{\mathfrak{g}}\big]_{\mathfrak{g}} + \big[Y, [Z, X]_{\mathfrak{g}}\big]_{\mathfrak{g}} + \big[Z, [X, Y]_{\mathfrak{g}}\big]_{\mathfrak{g}} = 0$$

for all $X, Y, Z \in \mathfrak{g}$. The basis elements T_a, $a = 1, \ldots, n$, for the Lie algebra are called **generators**, thus any element $X \in \mathfrak{g}$ can be written as $X = \vartheta^a T_a$, where a sum over the double index a is understood.

Any Lie group G possesses a well-defined Lie algebra $\mathfrak{g}$ which locally encodes the Lie group structure. Given a Lie group G, the vector space underlying the associated Lie algebra $\mathfrak{g}$ is the tangent space T_eG at the unit element. The identification $\mathrm{T}_eG \cong \mathfrak{g}$ can be made explicit as follows: Any Lie group G carries a distinguished 1-form, the **Maurer-Cartan form $\boldsymbol{\theta}$**, which contains the basic infinitesimal information about the structure of G. If $\ell_g : G \longrightarrow G$ is left multiplication with the element $g \in G$, a vector field $V \in \mathfrak{X}(G)$ is called **left-invariant** if $(\ell_g)_*V = V$ holds for all $g \in G$. Likewise, **right-invariant** vector fields are defined by invariance under the push-forward of right multiplication. The left-invariant $\mathfrak{g}$-valued 1-form θ on G is defined by

$$\begin{aligned} \theta_g &:= (\ell_{g^{-1}})_* : \mathrm{T}_gG \longrightarrow \mathrm{T}_eG = \mathfrak{g} \quad &&\text{(in general)} \\ &= g^{-1}\,\mathrm{d}g &&\text{(for matrix groups)} \end{aligned} \tag{A.4}$$

For any left-invariant vector field $V \in \mathfrak{X}(G)$, the Maurer-Cartan form satisfies $\theta(V) = V_e$, thus θ_e provides a natural identification of the tangent space T_eG and the Lie algebra $\mathfrak{g}$. The simple explicit form in the case of matrix groups allows to interpret the Maurer-Cartan form as the left logarithmic derivative of the identity map of G.

Any tangent vector of a Lie group can locally be extended to a left-invariant vector field in some neighborhood of the unit element. The commutator of two such vector fields is again a left-invariant vector field, which in turn gives a new vector in the tangent space at the unit element. This gives the natural description of the **Lie bracket of Lie groups** in explicit geometric form.

From the physicists point of view, the Lie algebra is much more important than the Lie group itself since it encodes the (anti-)commutativity of group elements near the unit element, which has direct effects for the uncertainty principle of quantum mechanics. Given a physical observable, the physicist usually knows the commutator of the associated operators—i.e. the physical information is encoded in the Lie bracket of the algebra.

A **Lie algebra homomorphism** $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ over Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ over the same field F is a F-linear map, such that the Lie bracket is preserved, i.e. $[f(x), f(y)]_{\mathfrak{h}} = f\big([x, y]_{\mathfrak{g}}\big)$ for all $x, y \in \mathfrak{g}$. A (linear) **representation of a Lie algebra** on the vector space V is a Lie algebra homomorphism $\tau : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) \cong \mathrm{End}(V)$, where $\mathfrak{gl}(V)$ denotes the Lie algebra of the Lie group $\mathrm{GL}(V)$. After a choice of basis for V, one can consider **Lie algebra matrix representations** analogous to Lie group matrix representations in terms of quadratic matrices.

A.18. The exponential mapping

The deep relation between Lie groups and algebras comes from the **exponential mapping**. For any Lie group G there is a local diffeomorphism

$$\exp : \mathfrak{g} \longrightarrow G$$

provided in a neighborhood of the unit element. If G is a compact group, the exponential mapping is always surjective onto the unit element's connection component, i.e. $\exp : \mathfrak{g} \twoheadrightarrow G^0$, where G^0 refers to the connection component of G containing the unit element. In general, however, $\exp : \mathfrak{g} \longrightarrow G^0$ is neither surjective nor injective.

Furthermore, the group structure is regained from the Lie algebra as follows: Let $g_1 = \mathrm{e}^X$ and $g_2 = \mathrm{e}^Y$ be two group elements from the image of the exponential mapping. The exponential for the product $g := g_1 g_2 = \mathrm{e}^Z$ is given by the **Campbell-Baker-Hausdorff formula**

$$Z = X + Y + \frac{1}{2}[X, Y]_{\mathfrak{g}} + \frac{1}{12}\big[X, [X, Y]_{\mathfrak{g}}\big]_{\mathfrak{g}} - \frac{1}{12}\big[Y, [X, Y]_{\mathfrak{g}}\big]_{\mathfrak{g}} + \ldots$$

which is completely determined by the Lie bracket. This effectively raises the group product into the Lie algebra. [Hal03, chp. 3] provides an excellent account on the mathematical details of this formula.

For matrix representations the exponential mapping is to be understood in terms of a formal power series of the respective matrices, whereas the definition in abstract terms requires additional techniques.[1] It is most important to realize, that Lie algebra and Lie group representations are not compatible in general. Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a linear representation of a Lie group G, then the push-forward $\rho_* : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ gives rise to an associated Lie algebra representation. However, the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\text{Lie algebra representation } \rho_*"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\not\circlearrowleft"] & \mathrm{End}(V) \cong \mathfrak{gl}(V) \arrow[d, "\exp"] \\
G \arrow[r, "\text{Lie group representation } \rho"'] & \mathrm{Aut}(V) \cong \mathrm{GL}(V)
\end{tikzcd}$$

does not commute in general. Only for a simply-connected Lie group G those compositions yield a commutative diagram, i.e. only for simply-connected Lie groups the Lie group and induced Lie algebra representations are compatible. This is of utmost importance in physics, as this incompatibility gives rise to tensorial and spinorial representations as neither the spatial rotation group SO(3) nor the Lorentz group $\mathcal{L}$ are simply connected. This will be investigated further in chap. 4 on spinors. The books [Che46], [Var98] and [HM98] shed more light on this intriguing issue.

The problem can be understood in much simpler terms: Consider the Lie groups SO(3) and SU(2), then the associated Lie algebras are $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$. One can prove the isomorphism $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ as Lie algebras, i.e. the groups SO(3) and SU(2) share the same structure at least locally in a neighborhood of the unit elements. From this point of view it is not really a

[1] For any given vector $X \in \mathfrak{g}$ there exists an associated unique integral curve $c_X :]-\varepsilon, \varepsilon[\longrightarrow G$ with $c_X(0) = e$ and $\frac{\mathrm{d}}{\mathrm{d}t} c_X(t)|_0 = X$, that follows the flow of the left-invariant vector field associated to X. The general exponential mapping is then defined by $\exp(X) := c_X(1)$.

surprise, that the Lie algebra—and thus the Lie algebra representations—do not fully contain the same information as the underlying Lie groups. In fact, it is the global group structure that is lost when just the Lie algebra is considered.[m]

A.19. Special representations

In particle physics one usually studies the irreducible (unitary) representations of certain symmetry groups. Some specific representations are always available:

- **Trivial representation**: To every element $g \in G$ of an arbitrary group assign the (unitary) 1×1-unit matrix, i.e.
$$\begin{aligned}\rho^{\mathbb{1}} : G &\longrightarrow \mathrm{U}(1)\\ g &\mapsto \mathbb{1}\,.\end{aligned}$$
Obviously the trivial representation is neither an isomorphism nor faithful, except in case of the trivial group $G = \{e\}$.
- **Fundamental representation**: The fundamental representation refers to the smallest-dimensional faithful[n] representation. In particular, for a matrix Lie group $G \subset \mathrm{GL}(V)$, where V is of minimal dimension, this representation is canonically given as
$$\begin{aligned}\rho^{\mathrm{Id}} : G &\hookrightarrow \mathrm{GL}(V)\\ g &\mapsto g.\end{aligned}$$
- **Complex conjugate representation**: Let ρ be a representation on the complex vector space V, then define
$$\begin{aligned}\bar{\rho} : G &\longrightarrow \mathrm{GL}(V)\\ g &\mapsto \overline{\rho(g)}\end{aligned}$$
as the (complex) conjugate representation, which is to be understood as taking the complex conjugated coefficients in the corresponding matrices. Without further reference, the conjugate representation is usually understood to be the complex conjugate representation of the fundamental representation.
- **Dual representation**: To any representation ρ on the vector space V there exists a corresponding dual representation ρ^* on the dual vector space V^*, which is given by taking the transpose in the corresponding matrix representation. In particular, for unitary representations the associated dual and complex conjugate representation are equivalent.
- **Adjoint representation**: For any Lie group G there is a natural automorphism provided by the **conjugation** or **adjoint mapping**
$$\begin{aligned}\mathrm{Ad} : G &\longrightarrow \mathrm{Aut}(G)\\ g &\mapsto \mathrm{Ad}_g\end{aligned} \qquad \text{where} \qquad \begin{aligned}\mathrm{Ad}_g : G &\xrightarrow{\cong} G\\ h &\mapsto \mathrm{Ad}_g(h) := ghg^{-1}.\end{aligned}$$
Since $\mathrm{Aut}(G)$ is a Lie group again, one can consider the associated tangent mapping $\mathrm{T}_e(c_g) : \mathfrak{g} \longrightarrow \mathfrak{g}$ at the unit element, which in turn provides an automorphism of the Lie algebra $\mathfrak{g}$ for any $g \in G$. This gives rise to the **adjoint representation of the Lie group**
$$\begin{aligned}\mathrm{ad} : G &\longrightarrow \mathrm{Aut}(\mathfrak{g})\\ g &\mapsto \mathrm{ad}_g\end{aligned} \qquad \text{where} \qquad \begin{aligned}\mathrm{ad}_g : \mathfrak{g} &\xrightarrow{\cong} \mathfrak{g}\\ X &\mapsto \mathrm{ad}_g(X) := \big[\mathrm{T}_e(c_g)\big](X),\end{aligned}$$

[m] Most physical textbook do speak of representations of groups in the context of quantum mechanics and commutators. But the relevant physical structure for the uncertainty principle is encoded in the Lie algebra—not the Lie group—thus the physicists are actual considering representations of the Lie algebra. This will be elaborated further in the spin geometry chapter.

[n] In mathematics, however, the fundamental representations refer to the representations associated with the fundamental weights (see app. B) of a simply-connected compact Lie group.

which is the natural faithful representation of the Lie group G on its Lie algebra $\mathfrak{g}$. Since $\dim V = \dim \mathfrak{g} = \dim G$ the dimension of the representation equals the dimension of the group manifold.

The adjoint representation of a Lie group is of particular importance in gauge theory, see chap. 2. Due to $\mathrm{T}_{\mathrm{Id}} \operatorname{Aut}(\mathfrak{g}) = \operatorname{End}(\mathfrak{g})$, the tangent map of the group's adjoint representation is

$$\begin{aligned} \widetilde{\mathrm{ad}} : \mathfrak{g} &\longrightarrow \operatorname{End}(\mathfrak{g}) \\ X &\mapsto \widetilde{\mathrm{ad}}_X \end{aligned} \qquad \text{where} \qquad \begin{aligned} \widetilde{\mathrm{ad}}_X : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ Y &\mapsto \widetilde{\mathrm{ad}}_X(Y) := [X, Y]_{\mathfrak{g}}, \end{aligned}$$

and provides the **adjoint representation of the Lie algebra**.

A.20. Real, complex and quaternionic representations

It is important to distinguish between the different (skew-)fields underlying a given representation, i.e. whether a representation is of real, complex or quaternionic type. It is not sufficient to just know the type of the representation space, since as modules (not vector spaces, since $\mathbb{H}$ is not a field) $\mathbb{R}^{4n} \cong \mathbb{C}^{2n} \cong \mathbb{H}^n$ holds.

In the strict mathematical sense, any representation ρ is at first considered to be of complex type. However, if the representation ρ is equivalent to the complex conjugate representation $\bar{\rho}$, i.e. $\rho \cong \bar{\rho}$, it is either of real or quaternionic type. If such a self-conjugated representation can be expressed in terms of square matrices with real or quaternionic coefficients, this gives the type of the representation. To summarize, a representation is called

- **complex**, if $\rho \ncong \bar{\rho}$, i.e. the representation is not equivalent to the complex conjugate representation. Thus, the representation can be expressed as
$$\rho : G \longrightarrow \mathrm{GL}_{\mathbb{C}}(V) \cong \mathrm{GL}(n; \mathbb{C}) \qquad \text{using } V \cong \mathbb{C}^n.$$
- **real**, if $\rho \cong \bar{\rho}$ and the representation can be expressed as
$$\rho : G \longrightarrow \mathrm{GL}_{\mathbb{R}}(V) \cong \mathrm{GL}(n; \mathbb{R}) \qquad \text{using } V \cong \mathbb{R}^n.$$
- **quaternionic**, if $\rho \cong \bar{\rho}$ and the representation cannot be expressed as a real representation. This implies, that is can be written as
$$\rho : G \longrightarrow \mathrm{GL}_{\mathbb{H}}(V) \cong \mathrm{GL}(n; \mathbb{H}) \qquad \text{using } V \cong \mathbb{H}^n.$$

The real and quaternionic representations can also be understood due to the existence of an additional structure, which is a specific mapping $J : V \longrightarrow V$ with square $J^2 = \mathrm{Id}$ in the real case and $J^2 = -\mathrm{Id}$ in the quaternionic case. This is called a real or quaternionic structure, see [BtD85, §II.6].

APPENDIX B

Roots, weights and lattices

In this chapter roots and weights are introduced, which are important tools in the general theory of representations. However, the sole purpose here is to provide those elementary notions, which are needed in order to understand the contents of the main sections, where the properties of the groups SO(10), G_2 and E_8 are used. Those groups—or rather their root lattices—are needed in the construction of the orbifold model outlined in chap. 10.

B.1. Weights

In short, weights are certain eigenvectors associated to the commutative part of either a Lie group or a Lie algebra, which can be associated to particular representations.

Let $\mathfrak{g}$ be a Lie algebra, V be a representation of $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ be a maximal commutative Lie subalgebra, which is also called the **Cartan subalgebra**. The **rank** of a Lie algebra is defined as the dimension of the Cartan subalgebra $\mathfrak{h}$. A weight is defined to be a $\mathbb{C}$-linear mapping $\lambda : \mathfrak{h} \longrightarrow \mathbb{C}$. To any such weight $\lambda \in \mathfrak{h}^*$ there is an associated **weight space** $V_\lambda \subset V$ defined by

$$V_\lambda := \big\{ v \in V : h \cdot v = \lambda(h) v \text{ for all } h \in \mathfrak{h} \big\},$$

whose nonzero elements are called **weight vectors**. A weight $\lambda \in \mathfrak{h}^*$ is called a **weight of the representation** V, if the corresponding weight space V_λ is nonzero. One of the great benefits of this construction is, that for any semi-simple Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ the finite-dimensional representation space V can be decomposed as $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, i.e. in a direct sum of all weight spaces associated to weights of this representation.

For a Lie group G with representation V the same construction is carried out using the maximal commutative (Lie) subgroup $H \subset G$ (often called the **maximal torus** in the case of a compact group) and taking differentials. The **rank** of a Lie group corresponds to the dimension of H. A homomorphism

$$\theta : H \longrightarrow \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$$

from the maximal commutative subgroup into the multiplicative group of complex numbers is called a **character**. A weight λ is then defined to be a differential of a character, evaluated at the unit element, i.e.

$$\lambda := \mathrm{T}\theta_e : \mathrm{T}_e H \cong \mathfrak{h} \longrightarrow \mathrm{T}_{\theta(e)} \mathbb{C}^\times \cong \mathbb{C},$$

such that again $\lambda \in \mathfrak{h}^*$. Conversely, let $\exp(\lambda) : H \longrightarrow \mathbb{C}^\times$ denote a character, such that $\lambda = \mathrm{T}\big[\exp(\lambda)\big]_e$ holds, then the **weight space** $V_\lambda \subset V$ of a weight λ is given as

$$V_\lambda := \big\{ v \in V : h \cdot v = \big[\exp(\lambda)\big](h) v \big\}.$$

Again, the elements of this space are called **weight vectors**, and the finite-dimensional representation space of a semi-simple Lie group G can also be decomposed into the corresponding weight spaces.

B.2. Roots

In the context of (compact) Lie group representation theory, the roots are simply the weights of the adjoint representation that exists for any Lie group. However, there is a deeper geometry behind those roots, which is involved in string theory in the form of momenta and compactification lattices, etc.

Let W be a finite-dimensional Euclidean space with inner product $\langle.,.\rangle$. For any vector $v \in W$, the **reflection** at the hyperplane perpendicular to v is given by the mapping

$$\sigma_v : W \xrightarrow{\cong} W$$
$$w \mapsto \sigma_v(w) := w - 2\frac{\langle v, w\rangle}{\|v\|^2}v,$$

which obviously satisfies idempotency, i.e. $\sigma_v \circ \sigma_v = \mathrm{Id}$. In general, a **root system** in W is a finite set $R = \{\alpha_1, \dots, \alpha_n\}$ of vectors $\alpha_i \in W$, called **roots**, that satisfies the following properties:

(1) $\operatorname{span} R = W$, i.e. the root system R is a generating system for the space W and $0 \notin R$.
(2) If $\alpha, \beta \in R$ are two proportional roots, then either $\alpha = \beta$ or $\alpha = -\beta$, i.e. the only allowed multiples are ± 1.
(3) The root system is closed under reflection at the hyperplanes orthogonal to any of the roots, i.e. for any two roots $\alpha, \beta \in R$ it follows $\sigma_\alpha(\beta) \in R$.
(4) For any two roots $\alpha, \beta \in R$ the reflection $\sigma_\alpha(\beta)$ and the root β differ by an integral multiple of α, i.e. $\sigma_\alpha(\beta) - \beta \in \mathbb{Z}\alpha$.

A more geometric interpretation of the last condition is, that the projection of the root β onto the line through the root α is in fact a half-integral multiple of α.

The **rank** of a root system $R \subset W$ is defined to be the dimension of W. A subset $S \subset R$ which is a basis of the space W is called a **system of simple roots**, if any root $\beta \in W$ may be written as

$$\beta = \sum_{\alpha \in S} n_\alpha \alpha,$$

with integral coefficients $n_\alpha \in \mathbb{Z}$, such that either all $n_\alpha \geq 0$ or $n_\alpha \leq 0$. Thus, a system of simple roots provides both a basis for the space W and (restricted to certain integral coefficients) for the root system R. Furthermore, for any given root system R, a subset $R^+ \subset R$ of **positive roots** can be chosen, such that the following properties are satisfied:

(1) For each root $\alpha \in R$ exactly one of the two possible multiples $\pm\alpha$ belongs to R^+, i.e. one root of each pair is distinguished to be the positive root.
(2) For any two positive roots $\alpha, \beta \in R^+$ where the sum $\alpha + \beta$ is also a root, it follows $\alpha + \beta \in R^+$, i.e. the positive roots are closed under addition in a restricted sense.

It can be shown, that for each system of simple roots there exists a corresponding choice of positive roots and vice versa, i.e. both notions are in fact equivalent.

B.3. Dynkin diagrams

Any two roots in a given root system R meet at an angle of 30, 45, 60, 90, 120, 135, 150 or 180 degrees. For a choice of simple roots, the angles 90, 120, 135 and 150 degrees are sufficient to describe the entire root system, if the ± 1-multiples and reflections are added. Given a root system $R \subset W$, there is an elegant way to depict the geometric properties of the simple roots $S \subset R$ in terms of Dynkin diagrams, which is independent of the particular choice of the simple roots. A **Dynkin diagram** is described as follows:

- For each simple root draw a **node**, i.e. the number of nodes gives the number of simple roots and thus the dimension of the space W.
- Between any pair of non-orthogonal simple roots an **edge** is drawn depending on the relative angle:
  - an undirected single edge indicates a 120 degree angle,
  - a directed double edge if they make an angle of 135 degrees,
  - a directed triple edge for 150 degree angles.

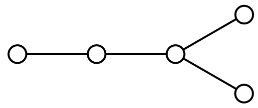


FIGURE B.1. The Dynkin diagram for SO(10), which is of the general type D_5.

In case of a double or triple edge, one of the roots is in fact shorter (with respect to the Euclidean norm induced by the inner product) and an arrow is pointing toward the shorter root.

The problem of classifying root systems is then equivalent to classifying Dynkin diagrams, which in turn is a rather simple problem of graph theory. One is left with four infinite series A_n, B_n, C_n, D_n of Dynkin diagrams and five exceptional cases E_6, E_7, E_8, F_4, G_2, where the lower index denotes the number of nodes (simple roots) in the diagram, i.e. the rank of the root system. Further material on Dynkin diagrams and root systems is found in [FH91, lect. 21].

B.4. The classical groups SO($2n$)

The representation theory and derivation of the root system for the classical groups is found in [BtD85, §V.6]. The special orthogonal groups SO($2n$) in even dimensions have a root system of the general type D_n for $n \geq 4$, which can be simply described by taking all integer vectors in $W = \mathbb{R}^n$ of length $\sqrt{2}$, i.e.

$$R_{\mathrm{SO}(2n)} = \left\{\beta_i \in \mathbb{Z}^n : \|\beta_i\| = \sqrt{2}\right\} \subset \mathbb{R}^n = W,$$

containing $2n(n-1)$ roots. A suitable choice of simple roots is provided via

$$\begin{aligned} \alpha_i &:= e_i - e_{i+1} \quad \text{for } 1 \leq i < n \\ \alpha_n &:= e_n + e_{n-1}, \end{aligned}$$

which in the special case of $n = 5$ for the group SO(10) can be written in terms of rows in the matrix

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ \hline 0 & 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 \\ \hline 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The Dynkin diagram of corresponding to those simple roots is depicted in fig. 1. Obviously, the groups SO($2n$) have rank n and the most important representations are of the following type:

$V = \mathbb{R}^{2n}$	$2n$-dim.	real	fundamental representation
$V = \mathfrak{so}(2n)$	$n(2n-1)$-dim.	real	adjoint representation

Note that the fundamental representation is induced directly in terms of matrices, as the SO($2n$) groups are matrix Lie groups.

B.5. The classical groups SU(n)

The special unitary groups SU(n) have a root system of type A_{n-1} for $n \geq 1$, which contains $n(n-1)$ roots spanning a $n-1$ dimensional Euclidean space. However, the roots are usually written as n-dimensional vectors with the additional condition that the components sum up to zero, i.e. the root α satisfies $\sum_{i=1}^n \alpha_i = 0$. The root system can then be specified as

$$R_{\mathrm{SU}(n)} = \left\{\beta_i \in \mathbb{Z}^n : \|\beta_i\| = \sqrt{2} \text{ and } \sum_{k=1}^{n} (\beta_i)_k = 0\right\},$$

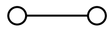


FIGURE B.2. The Dynkin diagram for SU(3), which is of type A_2. In general, a group SU(n) is of the general type A_{n-1}.

spanning a $n-1$-dimensional subspace W of $\mathbb{R}^n$. The roots are most conveniently written as

$$\Big(\underline{1,-1,0,0,\ldots,0}\Big),$$

where the underline indicates all possible permutations. A suitable choice of simple roots is provided by taking the "diagonal", i.e.

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = \left(\begin{array}{cccccc} 1 & -1 & 0 & \ldots & 0 & 0 \\ \hline 0 & 1 & -1 & \ldots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \ldots & -1 & 0 \\ \hline 0 & 0 & 0 & \ldots & 1 & -1 \end{array}\right)$$

The special case SU(3) is shown in the Dynkin diagram in fig. 2. Obviously, $\operatorname{rank}\mathrm{SU}(n) = n-1$, and the important representations are as follows:

$V = \mathbb{C}^n \cong \mathbb{R}^{2n}$	$2n$-dim.	complex	fundamental representation
$V = \mathfrak{su}(n)$	n^2-1-dim.	real	adjoint representation,

and the fundamental representation can be directly formulated in terms of unitary matrices satisfying $U^\dagger U = UU^\dagger = 1$.

B.6. The exceptional group E_8

The description of the exceptional groups is a rather complicated issue, and is best understood in terms of the octonions $\mathbb{O}$. This is a further extension of the quaternions, where in addition to commutativity the associativity of the multiplication is also lost. In terms of real vector spaces $\mathbb{O} \cong \mathbb{H}^2 \cong \mathbb{C}^4 \cong \mathbb{R}^8$ holds. An exhaustive account on octonions and their usage in both mathematics and physics is found in [Bae02], where the group E_8 is constructed as the natural isometry group of the 128-dimensional octo-octonionic projective plane $(\mathbb{O}\otimes\mathbb{O})\mathcal{P}^2$, which—at least formally—is constructed in a similar fashion as the real or complex projective spaces.

The group E_8 has rank 8 and the root system in $W = \mathbb{R}^8$ is described by taking all vectors β of length $\sqrt{2}$ such that the 8 components of 2β are all even or odd integers and the sum of all components is even, i.e.

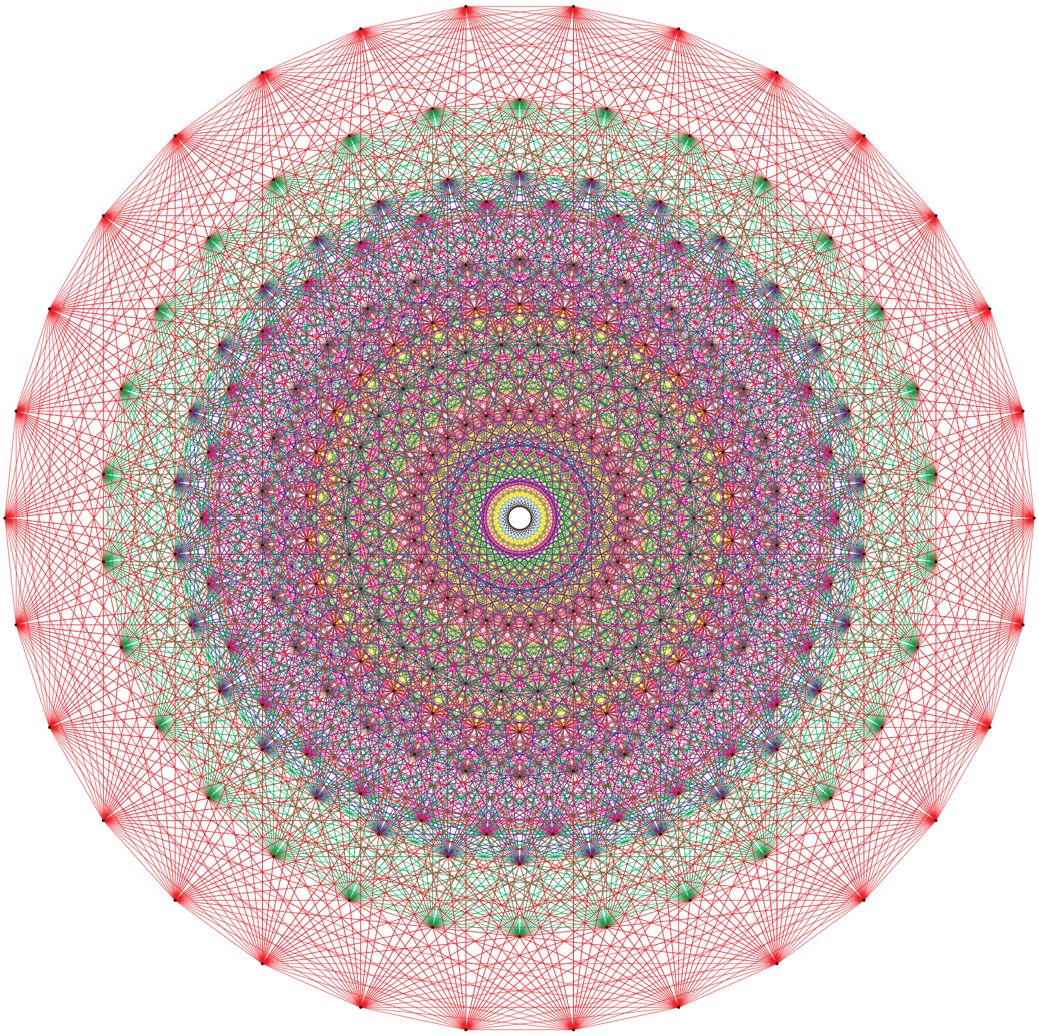
$$R_{\mathrm{E}_8} = \left(\beta \in \mathbb{Z}^8 \cup (\mathbb{Z}+\tfrac{1}{2})^8 : \|\beta\| = \sqrt{2},\ \sum_{i=1}^{8} \beta_i \text{ even}\right).$$

This root system contains 240 roots, which can be written in a much more accessible way as

$$\begin{aligned} R_{\mathrm{E}_8} = &\Big\{ \big(\underline{\pm 1, \pm 1, 0, 0, 0, 0, 0, 0}\big) \text{ including all permutations} \Big\} \\ &\cup \Big\{ \big(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\big) \text{ with even number of } \pm \text{ signs} \Big\}, \end{aligned}$$

where there are 112 roots with integral components (first line) and 128 roots with half-integral components (second line). A suitable choice of simple roots is given by

$$\begin{aligned} &\alpha_1 := \big(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\big) \\ &\alpha_i := e_i - e_{i-1} \quad \text{for } 2 \le i \le 7 \\ &\alpha_8 := e_7 + e_8, \end{aligned}$$



(Image by John Stembridge, based on a drawing by Peter McMullen.)

FIGURE B.3. The complexity and symmetry of the group E_8 is reflected in the full 8-dimensional E_8 root system, which is was projected to two dimensions. Each root is connected by a line to its 56 nearest neighbors, however, the lines from the origin to each root are not shown. Any two adjacent roots and the origin make an equilateral triangle in eight dimensions.

which corresponds to the rows of the simple root matrix

$$\left(\begin{array}{cccccccc}
+\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} \\
\hline
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right),$$

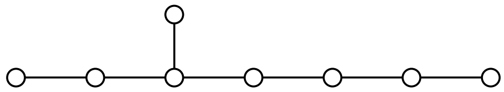


FIGURE B.4. The Dynkin diagram for E_8.

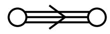


FIGURE B.5. The Dynkin diagram for G_2.

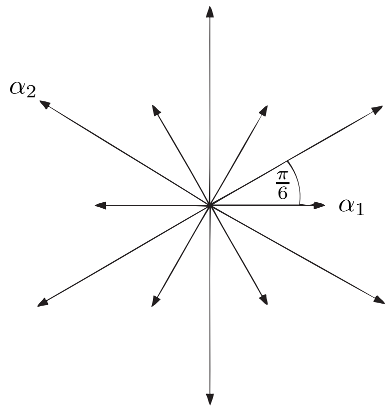


FIGURE B.6. The roots of the group G_2, with both simple roots α_1, α_2 indicated.

as depicted in the Dynkin diagram in fig. 4. In terms of representations, the exceptional group E_8 is distinguished by the unique fact, that the adjoint representation is the smallest non-trivial representation and corresponds to the fundamental representation:

$$V = \mathfrak{e}_8 \quad \text{248-dim.} \quad \text{real} \quad \text{adjoint (fundamental) representation}$$

The root system of E_8 can be understood as a subset of the E_8 **root lattice**

$$\Lambda_{E_8} := \left\{ x \in \mathbb{Z}^8 \cup (\mathbb{Z} + \tfrac{1}{2})^8 : \sum_{i=1}^{8} x_i \text{ even} \right\},$$

which arises naturally in the theory of (uni-)modular forms, see [Ser73]. The root system then consists of the lattice vectors nearest to the origin, i.e.

$$R_{E_8} = \left\{ \alpha \in \Lambda_{E_8} : \|\alpha\| = \sqrt{2} \right\},$$

which justifies the name. There is another E_8 lattice isomorphic to Λ_{E_8} defined by

$$\Lambda'_{E_8} := \left\{ x \in (\mathbb{Z} + \tfrac{1}{2})^8 : \sum_{i=1}^{8} x_i \text{ odd} \right\},$$

however, in this case the root system is not just given as a subset. Since both lattices are isomorphic, this does not affect the compactification, as $T_{\Lambda'_{E_8}} = T_{\Lambda_{E_8}}$.

B.7. The exceptional group G_2

The 14-dimensional Lie group G_2 is the smallest of the five exceptional Lie groups. Furthermore, it is simply connected and compact. As a rank-2 group, the root system can be directly depicted in two dimensions, see fig. B.6. From the picture, it is obvious, that the root system can be described by

$$R_{G_2} := \left\{ \left(\cos(\tfrac{\pi}{3}k), \sin(\tfrac{\pi}{3}k)\right), \sqrt{3}\left(\cos(\tfrac{\pi}{3}k + \tfrac{\pi}{2}), \sin(\tfrac{\pi}{3}k + \tfrac{\pi}{2})\right) : k = 0, \ldots, 5 \right\},$$

and the two simple roots spanning the 2-plane are indicated in the picture. The corresponding Dynkin diagram is found in fig. 5.

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